Parameterised Proof Complexity

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Semantics And Syntax: A Legacy of Alan Turing
Logical Approaches to Barriers in Complexity II, March 2012

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### Proof Complexity

<table>
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<th>Hard</th>
<th>Easy</th>
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<tr>
<td>Classical</td>
<td>$2^{\Omega(n)}$</td>
<td>$n^{O(1)}$</td>
</tr>
<tr>
<td>Parameterised</td>
<td>$n^{\Omega(k)}$</td>
<td>$f(k)n^{O(1)}$</td>
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Informally

... plus an additional “natural” parameter $k$

where $f(.)$ is any function and assuming $k$ is “small”, i.e. $k \ll n$. 
Parameterised Proof System: formal definition

**Definition.** *(parameterised proof system Cook-Reckhow style)* Given a class of parameterised tautologies \( L \subseteq \Sigma_T^* \times \Sigma_P^* \), a proof system for \( L \) is a surjective poly-time function \( p : \Sigma^* \rightarrow L \).

**Definition.** *(fixed-parameter bounded proof system)* The proof system \( p \) is fixed-parameter bounded iff there is a constant \( \alpha \) and a function \( f \) such that for all \((y, z) \in L\) there is \( x \in \Sigma^* \) such that \((y, z) = p(x)\) and \(|x| \leq f(|z|) |y|^\alpha\).
Fixed-Parameter Tractability and Parameterised Complexity

**Definition.** A parameterised problem \( L \subseteq \Sigma^* \times \Sigma^* \) is **fixed-parameter tractable** if there is a constant \( \alpha \) and an algorithm to determine if \((x, z) \in L\) in time \( f(|z|)|x|^\alpha\), where \( f \) is an arbitrary function.

**Definition.** (parameterised reduction) An \( m \)-reduction of the form \((x, k) \mapsto (x', f(k))\), running in time \( g(k)|x|^\alpha\), where \( \alpha \) is a constant and \( f, g \) are recursive functions.

**Definition.** (Weighted CNF Satisfiability) Given a propositional formula in conjunctive normal form, is there a satisfying assignment of weight at most \( k \), i.e. with at most \( k \) variables set to true (\( k \) is the parameter)?

**W-hierarchy:** \( \text{FPT} \subseteq W[1] \subseteq W[2] \subseteq \ldots W[t] \subseteq \ldots \)

**Proposition.** If \( \text{FPT} \equiv W[2] \) then there is a fixed-parameter bounded proof system for the class of all weighted CNF contradictions.
Generic Parameterisation of Sets of Clauses

The parameter \( k \) is a bound on the number of propositional variables that can be set to true.

1. (Naive) Add extra axioms: \( \neg v_{i_1} \lor \neg v_{i_2} \lor \ldots \lor \neg v_{i_{k+1}} \) for \( 1 \leq i_1 < i_2 < \cdots < i_{k+1} \leq n \). Count only the extra axioms that are used in the refutation.

2. (Embedded) Add additional variables \( p_{ij}, 1 \leq i \leq n, 1 \leq j \leq k \), and clauses

\[
\neg v_i \lor \bigvee_{j=1}^{k} p_{ij} \quad \text{for } 1 \leq i \leq n
\]

\[
\neg p_{i_1j} \lor \neg p_{i_2j} \quad \text{for } 1 \leq i_1 < i_2 \leq n, \text{ and } 1 \leq j \leq k.
\]
Complexity Gap for Tree-like Resolution

**Theorem.** (Riis) Given a FO logical sentence $\psi$ which fails in all finite models, consider its translation into a propositional CNF contradiction $C_{\psi,n}$, where $n$ is the size of the finite model. Then either

1. there is a constant $a$ such that for every $n$, the contradiction $C_{\psi,n}$ has a tree-like resolution refutation not bigger than $n^b$, or

2. there is a positive constant $a$ such that for every $n$, every tree-like resolution refutation of $C_{\psi,n}$ is of size at least $2^{an}$.

*Furthermore, 2 holds if and only if $\psi$ has an infinite model.*
Parameterised Complexity Gap

**Theorem.** (D., Martin, Szeider) Given a FO logical sentence \( \psi \), which fails in all finite models but holds in some infinite model. Consider the translation of \( \psi \) into a parameterised unsatisfiable set of clauses \( C_{\psi,n,k} \), where \( n \) is the size of the finite model, and the parameter \( k \) is the maximum total number of tuples over all relation symbols in \( \psi \). Then either

1. there are constants \( \alpha \) and \( \beta \) such that for every \( n \), the parameterised contradiction \( C_{\psi,n,k} \) has a tree-like resolution refutation not bigger than \( \beta^k n^\alpha \), or

2. there is a constant \( \gamma \), \( 0 < \gamma \leq 1 \), such that for every \( n \), every tree-like resolution refutation of \( C_{\psi,n,k} \) is of size at least \( n^{k\gamma} \).

Furthermore, 1 holds if and only for every (infinite) model of \( \psi \), the induced hypergraph has a finite dominating set.
**Definition.** (by picture)
In *every* (infinite) model, we have that

Proof. (intuition)

1. In a (finite) universe of size $n$, a finite dominating set $D$ of size $\alpha$ can be witnessed by a Boolean decision tree of size $n^\alpha$.

2. For every element $x \notin D$, a finite number of queries $\beta$ suffices to get a positive answer (a variable set to true). $k$ such answers can be achieved by a subtree of size $\beta^k$. 

□
Complexity of Parameterised Pigeon-Hole Principle

The pigeon-hole principle $PHP_{n+1}^n$ is the following (contradictory) set of clauses:

$$\bigvee_{j=1}^{n} v_{ij} \quad \text{for } 1 \leq i \leq n+1$$

$$\neg v_{i_1 j} \lor \neg v_{i_2 j} \quad \text{for } 1 \leq i_1 < i_2 \leq n+1, \ \text{and} \ 1 \leq j \leq n.$$  

**Proposition.** The parameterised pigeon-hole principle is

1. hard for tree-like resolution under the naive parameterisation, and
2. easy for resolution under the embedded parameterisation.
**Complexity of Parameterised Pigeon-Hole Principle**

**Theorem.** (Beyersdorff, Galesi, Lauria and Razborov) *The parameterised pigeon-hole principle is hard for bounded-depth Frege under the naive parameterisation.*

**Proof.** Assume that there is a refutation that uses fewer that $n^{k/5}$ extra clauses. Hit it with an assignment, chosen uniformly at random, of $n - \sqrt{n}$ pigeons into holes. The probability that a fixed extra axiom is hit at fewer than $k/2$ variables is no bigger than

$$\binom{k+1}{k/2} \frac{n^{-k/2}}{\binom{n}{\sqrt{n}}} \leq 2^{k+1} n^{-k/4}.$$ 

An extra axiom, which is hit at least $k/2$ variables, survives (does not evaluate to true) with probability no bigger than $(n - k/2)^{-k/2}$. As $2^{k+1} n^{-k/4} + (n - k/2)^{-k/2} < n^{-k/5}$, there is an assignment under which the parameterised $PHP^\sqrt{n+1}$ reduces to the (ordinary) $PHP^\sqrt{n}$.

$\square$
Consider a random graph $G = (V, E)$ on $n$ vertices, and in which every edge appears independently with probability $n^{-(1+\varepsilon)\frac{2}{k-1}}$ (so that, with high probability, there is no clique of size $k$). The $k$-clique contradiction can be encoded as follows.

\[
\bigvee_{v=1}^{n} p_{iv} \quad \text{for} \ 1 \leq i \leq k
\]

\[
\neg p_{iu} \lor \neg p_{iv} \quad \text{for} \ 1 \leq i \leq k, \text{ and } 1 \leq u < v \leq n
\]

\[
\neg p_{iu} \lor \neg p_{jv} \quad \text{for} \ 1 \leq i \neq j \leq k, \text{ and } \{u, v\} \notin E.
\]

**Theorem.** (Beyersdorff, Galesi and Lauria) The parameterised $k$-clique contradiction requires a tree-like resolution refutation of size $n^{\Omega((1-\varepsilon)k)}$ with high probability.

**Conjecture.** The parameterised $k$-clique contradiction is hard for resolution (bounded-depth Frege?).
"Natural" Parameterisation: variant(s) of the PHP

(A version of) the pigeon-hole principle from $k + 1$ to $k$ with $n$ in between.

\[
\bigvee_{i' = 1}^{n} q_{ii'} \quad \text{for } 1 \leq i \leq k + 1 \\
\neg q_{i'i'} \lor \neg q_{i''i} \quad \text{for } 1 \leq i' < i'' \leq k + 1, \text{ and } 1 \leq i \leq n \\
\neg q_{ii''} \lor \bigvee_{j=1}^{k} r_{i'j} \quad \text{for } 1 \leq i \leq k + 1, \text{ and } 1 \leq i' \leq n \\
\neg r_{i'j} \lor \neg r_{i''j} \quad \text{for } 1 \leq i' < i'' \leq n, \text{ and } 1 \leq j \leq k.
\]

**Conjecture.** The parameterised contradiction above is hard for resolution (and thus separates resolution from Res$(2)$ in the parameterised world).
"Natural" Parameterisation: FO sentences

Conjecture. Given a FO logical sentence $\psi$, which fails in all finite models but holds in some infinite model. Consider the translation of $\psi$ into a parameterised unsatisfiable set of clauses $D_{\psi,n,k}$ (where $n$ is the size of the finite model), saying that $\psi$ holds in some subset of size $k$. Then the parameterised contradiction $D_{\psi,n,k}$ is hard for resolution.

Remark. (D. and Riis) If there was no restriction on the size of the subset, the (non-parameterised) contradiction $D_{\psi,n}$ is exponentially hard for resolution.
References

