Locality from Circuit Lower Bounds

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In this talk

- consider finite directed graphs $G = (V^G, E^G)$

- $p$ is a graph property, if the following is true:
  
  if $G \cong H$, then $G$ has property $p \iff H$ has property $p$

- $q$ is a $k$-ary graph query, if the following is true:
  
  if $\pi : G \cong H$, then for all $a_1, \ldots, a_k \in V^G$,
  
  $(a_1, \ldots, a_k) \in q(G) \iff (\pi(a_1), \ldots, \pi(a_k)) \in q(H)$

- i.e., graph properties and queries are closed under isomorphisms.
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Logics expressing graph properties and queries

Classical logics like, e.g.

- FO (first-order logic: Boolean combinations + quantification over nodes)
- LFP (least fixed point logic: FO + inductive definitions of relations)

express graph properties and queries in a straightforward way.

**Example:**

\[ q(G) = \{ x \in V^G : x \text{ lies on a triangle} \} \]

is expressed in FO via

\[ \varphi(x) := \exists y \exists z \ ( E(x, y) \land E(y, z) \land E(z, x) ) \]

**Drawback:**

FO and LFP are too weak to express (some) computationally easy properties, e.g., properties concerning the size of \( V^G \) or \( E^G \).
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Overview

Introduction

Invariant logics

Non-expressibility results for Arb-invariant FO

Final Remarks
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Invariant logics

Idea:

- Extend the expressive power of a logic by allowing formulas to also use arithmetic predicates like <, +, ×, ... on $V^G$.

- For this, identify $V^G$ with the set $[n] := \{0, 1, \ldots, n-1\}$ for $n = |V^G|$ and interpret <, +, ×, ... in the natural way.

- To ensure closure under isomorphisms, restrict attention to formulas independent of the particular way of identifying $V^G$ with $[n]$. These formulas are called Arb-invariant.

Definition:

A -formula $\varphi(\vec{x})$ is -invariant on $G = (V^G, E^G)$ $\iff$ for all nodes $\bar{a}$ in $V^G$ and all linear orders $\prec_1$ and $\prec_2$ on $V^G$, ...
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**Definition:**

A $\phi(\vec{x})$ is $\invariant$ on $G = (V^G, E^G)$ if for all nodes $\vec{a}$ in $V^G$ and all linear orders $\prec_1$ and $\prec_2$ on $V^G$, $\phi(\vec{a})$ holds for both linear orders.
Invariant logics

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- Extend the expressive power of a logic by allowing formulas to also use arithmetic predicates like $<$, $+$, $\times$, ... on $V^G$.
- For this, identify $V^G$ with the set $[n] := \{0, 1, \ldots, n-1\}$ for $n = |V^G|$ and interpret $<$, $+$, $\times$, ... in the natural way.
- To ensure closure under isomorphisms, restrict attention to formulas independent of the particular way of identifying $V^G$ with $[n]$. These formulas are called Arb-invariant.

Definition:

A FO$(E, \prec)$-formula $\varphi(\bar{x})$ is order-invariant on $G = (V^G, E^G)$ if for all nodes $\bar{a}$ in $V^G$ and all linear orders $\prec_1$ and $\prec_2$ on $V^G$,

$$(G, \prec_1) \models \varphi(\bar{a}) \iff (G, \prec_2) \models \varphi(\bar{a}).$$
Invariant logics

**Idea:**

- Extend the expressive power of a logic by allowing formulas to also use arithmetic predicates like \(<, +, \times, \ldots\) on \(V^G\).

- For this, identify \(V^G\) with the set \([n] := \{0, 1, \ldots, n-1\}\) for \(n = |V^G|\) and interpret \(<, +, \times, \ldots\) in the natural way.

- To ensure closure under isomorphisms, restrict attention to formulas independent of the particular way of identifying \(V^G\) with \([n]\). These formulas are called Arb-invariant.

**Definition:**

A FO\((E, <, +)\)-formula \(\varphi(\vec{x})\) is addition-invariant on \(G = (V^G, E^G)\) if and only if for all nodes \(\vec{a}\) in \(V^G\) and all linear orders \(<_1\) and \(<_2\) on \(V^G\), and the matching addition relations \(+_1, +_2\),

\[
(G, <_1, +_1) \models \varphi(\vec{a}) \iff (G, <_2, +_2) \models \varphi(\vec{a}).
\]
Invariant logics

Idea:

- Extend the expressive power of a logic by allowing formulas to also use arithmetic predicates like $<, +, \times, \ldots$ on $V^G$.
- For this, identify $V^G$ with the set $[n] := \{0, 1, \ldots, n-1\}$ for $n = |V^G|$ and interpret $<, +, \times, \ldots$ in the natural way.
- To ensure closure under isomorphisms, restrict attention to formulas independent of the particular way of identifying $V^G$ with $[n]$. These formulas are called Arb-invariant.

Definition:

A FO$(E, \prec, +, \times)$-formula $\varphi(\vec{x})$ is $(+, \times)$-invariant on $G = (V^G, E^G)$ if for all nodes $\vec{a}$ in $V^G$ and all linear orders $\prec_1$ and $\prec_2$ on $V^G$, and the matching addition relations $+_1, +_2$, and the according multiplications $\times_1, \times_2$,

\[
(G, \prec_1, +_1, \times_1) \models \varphi(\vec{a}) \iff (G, \prec_2, +_2, \times_2) \models \varphi(\vec{a}).
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Invariant logics

Idea:

- Extend the expressive power of a logic by allowing formulas to also use arithmetic predicates like $<, +, \times, \ldots$ on $V^G$.

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- To ensure closure under isomorphisms, restrict attention to formulas independent of the particular way of identifying $V^G$ with $[n]$. These formulas are called Arb-invariant.

Definition:

A FO($E, <, +, \times, \ldots$)-formula $\varphi(\vec{x})$ is Arb-invariant on $G = (V^G, E^G)$ $\iff$ for all nodes $\vec{a}$ in $V^G$ and all linear orders $\prec_1$ and $\prec_2$ on $V^G$, and the matching addition relations $+_1, +_2$, and the according multiplications $\times_1, \times_2$, and other numerical predicates,

$$ (G, \prec_1, +_1, \times_1, \ldots) \models \varphi(\vec{a}) \iff (G, \prec_2, +_2, \times_2, \ldots) \models \varphi(\vec{a}). $$
Example

- An **addition-invariant** FO\((E, \prec, +)\)-sentence \(\varphi\) such that
  \[
  G \models \varphi \iff |V^G| \text{ is odd.}
  \]

\[
\varphi := \exists x \exists z \left( x + x = z \land \forall y \left( y \prec z \lor y = z \right) \right)
\]

- Similarly, there is an **\((+, \times)\)-invariant** FO\((E, \prec, +, \times)\)-sentence \(\psi\) such that
  \[
  G \models \psi \iff |V^G| \text{ is a prime number.}
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Thus:

order-invariant FO \(\prec\) addition-invariant FO \(\prec\) Arb-invariant FO.
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- Similarly, there is an ($+, \times$)-invariant FO($E, \prec, +, \times$)-sentence $\psi$ such that

  $G \models \psi \iff |V^G|$ is a prime number.

Thus:

$FO < \text{order-invariant FO} < \text{addition-invariant FO} < \text{Arb-invariant FO}$. 
Theorem:

There is a finite relational signature $\tau$ which, among other symbols, contains symbols $U$ (unary) and $E$ (binary), such that the following is true:

There is a FO-definable class $\mathcal{C}$ of finite $\tau$-structures such that connectivity of $(U, E|_U)$ is definable in order-invariant FO, but not in FO on $\mathcal{C}$.

Here, connectivity of $(U, E|_U)$ on $\mathcal{C}$ means:
Decide, for a given $\tau$-structure $\mathcal{A}$ in $\mathcal{C}$, if the graph $G = (U^\mathcal{A}, E^\mathcal{A}|_U)$ is connected.
Expressive power of invariant logics

**Known results:**

- **Order-invariant LFP** precisely captures the polynomial time computable graph properties and queries. (Immerman, Vardi, 1982)

- **Arb-invariant LFP** precisely captures the graph properties and queries that belong to the complexity class $P_{/poly}$. (Makowsky, 1998)

  $P_{/poly}$ consists of all problems solvable by circuit families of polynomial size.

- **Arb-invariant FO** precisely captures the graph properties and queries that belong to the circuit complexity class $AC^0$.

  $AC^0$ consists of all problems solvable by circuit families of polynomial size and constant depth.

- **$(+ , \times)$-invariant FO** precisely captures the graph properties and queries that belong to uniform $AC^0$. 
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  \(AC^0\) consists of all problems solvable by circuit families of polynomial size and constant depth.

- \((+, \times)\)-invariant FO precisely captures the graph properties and queries that belong to uniform \(AC^0\).
Invariant logics are not logics in the strict formal sense: They have an undecidable syntax.

Precisely:

The following problem is undecidable (by reduction from Trakhtenbrot’s theorem)

**ORDER-INVARINACE ON FINITE GRAPHS:**

*Input:* an FO($E, \prec$)-sentence $\varphi$

*Question:* Is $\varphi$ order-invariant on all finite graphs?
Overview

Introduction

Invariant logics

Non-expressibility results for Arb-invariant FO

Final Remarks
Non-expressibility results for Arb-invariant FO

**Known results:**

- The existence of a $k$-clique cannot be expressed in Arb-invariant FO sentence using only $\lfloor k/4 \rfloor$ variables. (Rossman’08)

- The query $\text{Reach}_{f(n)}(x, y)$, selecting all pairs of nodes connected by a path of length $\leq f(n)$, is not definable in Arb-invariant FO. (Ajtai’89: for any unbounded function $f$; already on graphs of bounded degree)

- The class of graphs having an **even number of edges** is not definable in Arb-invariant monadic existential second-order logic. (Ajtai’83)
Locality of queries — Neighborhoods

Graph $G = (V, E)$

Distance $\text{dist}(u, v)$: length of a shortest path between $u, v$ in $G$.

Ball $N_r(a)$ of radius $r$ at $a$ in $G$.

Neighborhood $\mathcal{N}_r(a)$ of radius $r$ at $a$ in $G$. 
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![Graph with nodes and edges representing neighborhoods](image.png)
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![Graph with labeled vertices $u$ and $v$ and distances marked]
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![Diagram showing a graph with nodes and edges, illustrating the concepts of graph, distance, ball, and neighborhood.](image-url)
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![Graph with neighborhoods at point a and radii r = 0 and r = 1]
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![Graph and neighborhoods diagram]

$r = 2$

$r = 1$

$r = 0$
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![Diagram of a graph with labeled nodes and edges, illustrating the concept of neighborhoods and balls around a node.](image)
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Non-expressibility results for Arb-invariant FO

Final Remarks

Locality of queries — Neighborhoods

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![Graph with graph theory concepts]
Locality of queries — Neighborhoods

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Ball $N_r(a)$ of radius $r$ at $a$ in $G$.

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Local queries

- For a list \( a = a_1, \ldots, a_k \) of nodes, \( N_r^G(a) = N_r^G(a_1) \cup \cdots \cup N_r^G(a_k) \).
- The \( r \)-neighborhood \( N_r^G(a) \) is the structure \((G_{\mid N_r^G(a)}, a)\) consisting of the induced subgraph of \( G \) on \( N_r^G(a) \), together with the distinguished nodes \( a \).

**Definition:** Let \( q \) be a \( k \)-ary graph query. Let \( f : \mathbb{N} \to \mathbb{N} \).

\( q \) is called \( f(n) \)-local if there is an \( n_0 \) such that for every \( n \geq n_0 \) and every graph \( G \) with \( |V^G| = n \), the following is true for all \( k \)-tuples \( a \) and \( b \) of nodes:

\[
\text{if } N_r^G(a) \cong N_r^G(b) \text{ then } a \in q(G) \iff b \in q(G).
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Local queries

- For a list $a = a_1, \ldots, a_k$ of nodes, $N_r^G(a) = N_r^G(a_1) \cup \cdots \cup N_r^G(a_k)$.

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**Definition:** Let $q$ be a $k$-ary graph query. Let $f : \mathbb{N} \to \mathbb{N}$. $q$ is called $f(n)$-local if there is an $n_0$ such that for every $n \geq n_0$ and every graph $G$ with $|V^G| = n$, the following is true for all $k$-tuples $a$ and $b$ of nodes:

$$\text{if } N_{f(n)}^G(a) \cong N_{f(n)}^G(b) \text{ then } a \in q(G) \iff b \in q(G).$$
Locality of invariant FO

**Theorem:** \(\text{(Grohe, Schwentick '98)}\)

For every query \(q\) expressible by order-invariant FO there is a \(c \in \mathbb{N}\) such that \(q\) is \(c\)-local.

**Open Question:** Is addition-invariant FO \(c\)-local?

**Theorem:** \(\text{(Anderson, Melkebeek, S., Segoufin, '11)}\)

(a) For every query \(q\) expressible by Arb-invariant FO there is a \(c \in \mathbb{N}\) such that \(q\) is \((\log n)^c\)-local.

(b) For every \(d \in \mathbb{N}\) there is a \((+, \times)\)-invariant FO query that is not \((\log n)^d\)-local.
Locality of invariant FO

**Theorem:** (Grohe, Schwentick ’98)

For every query $q$ expressible by order-invariant FO there is a $c \in \mathbb{N}$ such that $q$ is $c$-local.

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(b) For every $d \in \mathbb{N}$ there is a $(+, \times)$-invariant FO query that is not $(\log n)^d$-local.
Use locality for proving non-expressibility

**Example:** The reachability query

\[
\text{REACH}(G) := \{ (a_1, a_2) : \text{there is a directed path from } a_1 \text{ to } a_2 \text{ in } G \}
\]

is not $n^5$-local and thus cannot be expressed in Arb-invariant FO.

**Proof:** Consider the graph $G$:

![Graph Diagram]

- Use locality for proving non-expressibility
- Example: The reachability query
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  is not $n^5$-local and thus cannot be expressed in Arb-invariant FO.
  
  **Proof:** Consider the graph $G$: (Diagram of a graph with nodes $a_1$, $b_1$, $a_2$, $b_2$)


Use locality for proving non-expressibility

Similarly, one obtains that the following queries are not definable in Arb-invariant FO:

- Does node $x$ lie on a cycle?
- Does node $x$ belong to a connected component that is acyclic?
- Is node $x$ reachable from a node that belongs to a triangle?
- Do nodes $x$ and $y$ have the same distance to node $z$?
Proof of locality theorem — upper bound (1/5)

(a) For every query $q$ expressible by Arb-invariant FO there is a $c \in \mathbb{N}$ such that $q$ is $(\log n)^c$-local.

Idea:

- Let $q$ be expressible by an Arb-invariant FO formula.
- Then, $q$ can be computed by an $\text{AC}^0$ circuit family $C$.
- Assume that $q$ is not $(\log n)^c$-local (for any $c \in \mathbb{N}$), and modify $C$ to obtain an $\text{AC}^0$ circuit family computing

$$\text{PARITY} := \{ w \in \{0, 1\}^* : |w|_1 \text{ is even} \}.$$

- This contradicts known lower bounds in circuit complexity theory (Håstad’86).
Proof of locality theorem — upper bound (2/5)

How to compute a graph query $q(x)$ by an $\text{AC}^0$ circuit family $C$?

- Represent graph $G = (V, E)$ by a bitstring $\beta(G)$ corresponding to an adjacency matrix for $G$.
- Represent a node $a \in V$ by the bitstring $\beta(a)$ of the form $0^*10^*$, carrying the 1 at position $i$ iff node $a$ corresponds to the $i$-th row/column of the adjacency matrix.
- Let $\text{Rep}(G, a)$ be the set of all bitstrings $\beta(G)\beta(a)$, corresponding to all adjacency matrices of $G$ (i.e., all ways of embedding $V$ in $\{1, \ldots, |V|\}$). Thus, $\text{Rep}(G, a)$ is the set of all bitstrings representing $(G, a)$.
- A unary graph query $q(x)$ is computed by a circuit family $C = (C_n)_{n \in \mathbb{N}}$ iff the following is true:
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- Known: A unary graph query $q(x)$ is definable in Arb-invariant FO $\iff$ it is computed by a circuit family of constant depth and polynomial size. (implicit in Immerman’87)
Proof of locality theorem — upper bound (2/5)

How to compute a graph query \( q(x) \) by an \( \text{AC}^0 \) circuit family \( C \)?

- Represent graph \( G = (V, E) \) by a bitstring \( \beta(G) \) corresponding to an adjacency matrix for \( G \).

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How to compute a graph query $q(x)$ by an $AC^0$ circuit family $C$?

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Proof of locality theorem — upper bound (3/5)

Let $q(x)$ be a unary graph query expressible in Arb-invariant FO. Let $C = (C_n)_{n \in \mathbb{N}}$ be a circuit family of constant depth $d$ and polynomial size $p(n)$ computing $q$. I.e., for all $G = (V, E), a \in V, \gamma \in \text{Rep}(G, a)$: $a \in q(G) \iff C|_{\gamma}$ accepts $\gamma$.

For contradiction, assume $q(x)$ is not $(\log n)^c$-local, for any $c \in \mathbb{N}$. Thus: For all $c, n_0$ there exist $n > n_0, G = (V, E)$ with $n$ nodes, $a, b \in V$ such that for $m := (\log n)^c, \mathcal{N}_m^G(a) \cong \mathcal{N}_m^G(b)$, but $a \in q(G)$ and $b \not\in q(G)$.

For simplicity, consider the special case that $\text{dist}(a, b) > 2m$.

**Key Lemma:**

Let $m \in \mathbb{N}, G = (V, E), a, b \in V$ such that $\mathcal{N}_m^G(a) \cong \mathcal{N}_m^G(b)$ and $\text{dist}(a, b) > 2m$. Let circuit $C$ accept all strings in $\text{Rep}(G, a)$ and reject all strings in $\text{Rep}(G, b)$.

Then there is a circuit $\tilde{C}$ of the same size & depth as $C$ computing parity on $m$ bits.

**Theorem:**

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There exist $\ell, m_0 > 0$ such that for all $m \geq m_0$, no circuit of depth $d$ and size $2^{\ell \cdot m^{1/(d-1)}}$ computes parity on $m$ bits.

Contradiction for $c = 2d$, since $2^{\ell \cdot m^{1/(d-1)}} > n^{\ell \log n} > p(n)$. \qed
Proof of locality theorem — upper bound \((3/5)\)

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Nicole Schweikardt

Locality from Circuit Lower Bounds

20/29
Proof of locality theorem — upper bound (3/5)

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Nicole Schweikardt

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NICOLE SCHWEIKARDT

LOCALITY FROM CIRCUIT LOWER BOUNDS
Proof of locality theorem — upper bound (3/5)

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Proof of locality theorem — upper bound (4/5)

**Key Lemma:**
Let $m \in \mathbb{N}$, $G = (V, E)$, $a, b \in V$ such that $N^G_m(a) \cong N^G_m(b)$ and $\text{dist}(a, b) > 2m$. Let circuit $C$ accept all strings in $\text{Rep}(G, a)$ and reject all strings in $\text{Rep}(G, b)$.

Then there is a circuit $\tilde{C}$ of the same size & depth as $C$ computing parity on $m$ bits.

**Proof:**

Consider $w \in \{0, 1\}^m$.

For $i \in \{0, 1, \ldots, m - 1\}$ with $w_i = 1$:

Swap the endpoints of the edges leaving $N_i(a)$ with the corresponding endpoints of the edges leaving $N_i(b)$.

The resulting graph $G_w \cong G$.

$$(G_w, a) \cong \begin{cases} (G, a), & \text{if } |w|_1 \text{ even} \\ (G, b), & \text{if } |w|_1 \text{ odd} \end{cases}$$

Circuit $C$ distinguishes these cases.
Proof of locality theorem — upper bound (4/5)

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Consider $w \in \{0, 1\}^m$.
For $i \in \{0, 1, \ldots, m - 1\}$ with $w_i = 1$:

Swap the endpoints of the edges leaving $N_i(a)$ with the corresponding endpoints of the edges leaving $N_i(b)$.

The resulting graph $G_w \cong G$.

$$(G_w, a) \cong \begin{cases} (G, a), & \text{if } |w|_1 \text{ even} \\ (G, b), & \text{if } |w|_1 \text{ odd} \end{cases}$$

Circuit $C$ distinguishes these cases.
Proof of locality theorem — upper bound (4/5)

**Key Lemma:**

Let $m \in \mathbb{N}$, $G = (V, E)$, $a, b \in V$ such that $\mathcal{N}_m^G(a) \cong \mathcal{N}_m^G(b)$ and $\text{dist}(a, b) > 2m$.

Let circuit $C$ accept all strings in $\text{Rep}(G, a)$ and reject all strings in $\text{Rep}(G, b)$.

Then there is a circuit $\tilde{C}$ of the same size & depth as $C$ computing parity on $m$ bits.

**Proof:**

Consider $w \in \{0, 1\}^m$.

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Proof of locality theorem — upper bound (4/5)

**Key Lemma:**

Let \( m \in \mathbb{N} \), \( G = (V, E) \), \( a, b \in V \) such that \( N^G_m(a) \cong N^G_m(b) \) and \( \text{dist}(a, b) > 2m \).

Let circuit \( C \) accept all strings in \( \text{Rep}(G, a) \) and reject all strings in \( \text{Rep}(G, b) \).

Then there is a circuit \( \tilde{C} \) of the same size & depth as \( C \) computing parity on \( m \) bits.

**Proof:**

Consider \( w \in \{0, 1\}^m \).

For \( i \in \{0, 1, \ldots, m-1\} \) with \( w_i = 1 \):

- Swap the endpoints of the edges leaving \( N_i(a) \) with the corresponding endpoints of the edges leaving \( N_i(b) \).

The resulting graph \( G_w \cong G \).

\[
(G_w, a) \cong \begin{cases} (G, a), & \text{if } |w_1| \text{ even} \\ (G, b), & \text{if } |w_1| \text{ odd} \end{cases}
\]

Circuit \( C \) distinguishes these cases.
Key Lemma:
Let \( m \in \mathbb{N} \), \( G = (V, E) \), \( a, b \in V \) such that \( N^G_m(a) \cong N^G_m(b) \) and \( \text{dist}(a, b) > 2m \).
Let circuit \( C \) accept all strings in \( \text{Rep}(G, a) \) and reject all strings in \( \text{Rep}(G, b) \).
Then there is a circuit \( \tilde{C} \) of the same size & depth as \( C \) computing parity on \( m \) bits.

Proof:
Consider \( w \in \{0, 1\}^m \).
For \( i \in \{0, 1, \ldots, m - 1\} \) with \( w_i = 1 \):
Swapping the endpoints of the edges leaving \( N_i(a) \) with the corresponding endpoints of the edges leaving \( N_i(b) \).
The resulting graph \( G_w \cong G \).

\[
(G_w, a) \cong \begin{cases} 
(G, a), & \text{if } |w|_1 \text{ even} \\
(G, b), & \text{if } |w|_1 \text{ odd}
\end{cases}
\]
Circuit \( C \) distinguishes these cases.
Proof of locality theorem — upper bound (4/5)

**Key Lemma:**
Let \( m \in \mathbb{N}, \ G = (V, E), \ a, b \in V \) such that \( N^G_m(a) \cong N^G_m(b) \) and \( \text{dist}(a, b) > 2m \). Let circuit \( C \) accept all strings in \( \text{Rep}(G, a) \) and reject all strings in \( \text{Rep}(G, b) \). Then there is a circuit \( \tilde{C} \) of the same size & depth as \( C \) computing parity on \( m \) bits.

**Proof:**
Consider \( w \in \{0, 1\}^m \).
For \( i \in \{0, 1, \ldots, m-1\} \) with \( w_i = 1 \):

- **Swap the endpoints of the edges** leaving \( N_i(a) \) with the corresponding endpoints of the edges leaving \( N_i(b) \).

The resulting graph \( G_w \cong G \).

\[
(G_w, a) \cong \begin{cases} (G, a), & \text{if } |w|_1 \text{ even} \\ (G, b), & \text{if } |w|_1 \text{ odd} \end{cases}
\]

Circuit \( C \) distinguishes these cases.
**Key Lemma:**

Let $m \in \mathbb{N}$, $G = (V, E)$, $a, b \in V$ such that $\mathcal{N}_m^G(a) \cong \mathcal{N}_m^G(b)$ and $\text{dist}(a, b) > 2m$. Let circuit $C$ accept all strings in $\text{Rep}(G, a)$ and reject all strings in $\text{Rep}(G, b)$. Then there is a circuit $\tilde{C}$ of the same size & depth as $C$ computing parity on $m$ bits.

**Proof:**

Consider $w \in \{0, 1\}^m$.

For $i \in \{0, 1, \ldots, m-1\}$ with $w_i = 1$:

Swap the endpoints of the edges leaving $N_i(a)$ with the corresponding endpoints of the edges leaving $N_i(b)$.

The resulting graph $G_w \cong G$.

$$(G_w, a) \cong \begin{cases} (G, a), & \text{if } |w|_1 \text{ even} \\ (G, b), & \text{if } |w|_1 \text{ odd} \end{cases}$$

Circuit $C$ distinguishes these cases.
**Key Lemma:**

Let $m \in \mathbb{N}$, $G = (V, E)$, $a, b \in V$ such that $\mathcal{N}_m^G(a) \cong \mathcal{N}_m^G(b)$ and $\text{dist}(a, b) > 2m$. Let circuit $C$ accept all strings in $\text{Rep}(G, a)$ and reject all strings in $\text{Rep}(G, b)$. Then there is a circuit $\tilde{C}$ of the same size & depth as $C$ computing parity on $m$ bits.

**Proof:**

Consider $w \in \{0, 1\}^m$.

For $i \in \{0, 1, \ldots, m - 1\}$ with $w_i = 1$:

*Swap the endpoints of the edges leaving $N_i(a)$ with the corresponding endpoints of the edges leaving $N_i(b)$.*

The resulting graph $G_w \cong G$.

$$(G_w, a) \cong \begin{cases} (G, a), & \text{if } |w|_1 \text{ even} \\ (G, b), & \text{if } |w|_1 \text{ odd} \end{cases}$$

Circuit $C$ distinguishes these cases.
**Key Lemma:**

Let \( m \in \mathbb{N} \), \( G = (V, E) \), \( a, b \in V \) such that \( N_m^G(a) \cong N_m^G(b) \) and \( \text{dist}(a, b) > 2m \). Let circuit \( C \) accept all strings in \( \text{Rep}(G, a) \) and reject all strings in \( \text{Rep}(G, b) \). Then there is a circuit \( \tilde{C} \) of the same size & depth as \( C \) computing parity on \( m \) bits.

**Proof:**

Consider \( w \in \{0, 1\}^m \).

For \( i \in \{0, 1, \ldots, m-1\} \) with \( w_i = 1 \):

Swap the endpoints of the edges leaving \( N_i(a) \) with the corresponding endpoints of the edges leaving \( N_i(b) \).

The resulting graph \( G_w \cong G \).

\[
(G_w, a) \cong \begin{cases} 
(G, a), & \text{if } |w|_1 \text{ even} \\
(G, b), & \text{if } |w|_1 \text{ odd}
\end{cases}
\]

Circuit \( C \) distinguishes these cases.
Proof of locality theorem — upper bound (4/5)

**Key Lemma:**

Let $m \in \mathbb{N}$, $G = (V, E)$, $a, b \in V$ such that $\mathcal{N}_m^G(a) \equiv \mathcal{N}_m^G(b)$ and $\text{dist}(a, b) > 2m$. Let circuit $C$ accept all strings in $\text{Rep}(G, a)$ and reject all strings in $\text{Rep}(G, b)$. Then there is a circuit $\tilde{C}$ of the same size & depth as $C$ computing parity on $m$ bits.

**Proof:**

Consider $w \in \{0, 1\}^m$.

For $i \in \{0, 1, \ldots, m-1\}$ with $w_i = 1$:  

*Swap the endpoints of the edges leaving $N_i(a)$ with the corresponding endpoints of the edges leaving $N_i(b)$.*

The resulting graph $G_w \equiv G$. 

$$(G_w, a) \equiv \begin{cases} 
(G, a), & \text{if } |w|_1 \text{ even} \\
(G, b), & \text{if } |w|_1 \text{ odd}
\end{cases}$$

Circuit $C$ distinguishes these cases.
**Key Lemma:**

Let $m \in \mathbb{N}$, $G = (V, E)$, $a, b \in V$ such that $\mathcal{N}_m^G(a) \cong \mathcal{N}_m^G(b)$ and $\text{dist}(a, b) > 2m$. Let circuit $C$ accept all strings in $\text{Rep}(G, a)$ and reject all strings in $\text{Rep}(G, b)$. Then there is a circuit $\tilde{C}$ of the same size & depth as $C$ computing parity on $m$ bits.

**Proof:**

Consider $w \in \{0, 1\}^m$.

For $i \in \{0, 1, \ldots, m-1\}$ with $w_i = 1$:

Swap the endpoints of the edges leaving $N_i(a)$ with the corresponding endpoints of the edges leaving $N_i(b)$.

The resulting graph $G_w \cong G$.

$$(G_w, a) \cong \begin{cases} (G, a), & \text{if } |w|_1 \text{ even} \\ (G, b), & \text{if } |w|_1 \text{ odd} \end{cases}$$

Circuit $C$ distinguishes these cases.
Key Lemma:

Let \( m \in \mathbb{N} \), \( G = (V, E) \), \( a, b \in V \) such that \( N_m^G(a) \cong N_m^G(b) \) and \( \text{dist}(a, b) > 2m \).

Let circuit \( C \) accept all strings in \( \text{Rep}(G, a) \) and reject all strings in \( \text{Rep}(G, b) \).

Then there is a circuit \( \tilde{C} \) of the same size & depth as \( C \) computing parity on \( m \) bits.

Proof:

Consider \( w \in \{0, 1\}^m \).

For \( i \in \{0, 1, \ldots, m-1\} \) with \( w_i = 1 \):

\[
\text{Swap the endpoints of the edges leaving } N_i(a) \text{ with the corresponding endpoints of the edges leaving } N_i(b).
\]

The resulting graph \( G_w \cong G \).

\[
(G_w, a) \cong \begin{cases} (G, a), & \text{if } |w|_1 \text{ even} \\ (G, b), & \text{if } |w|_1 \text{ odd} \end{cases}
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Circuit $C$ distinguishes these cases.
Proof of locality theorem — upper bound (5/5)

**Key Lemma:**

Let \( m \in \mathbb{N}, \ G = (V, E), \ a, b \in V \) such that \( \mathcal{N}_m^G(a) \cong \mathcal{N}_m^G(b) \) and \( \text{dist}(a, b) > 2m \).

Let circuit \( C \) accept all strings in \( \text{Rep}(G, a) \) and reject all strings in \( \text{Rep}(G, b) \).

Then there is a circuit \( \tilde{C} \) of the same size & depth as \( C \) computing parity on \( m \) bits.

How to obtain \( \tilde{C} \) from \( C \)?

- Consider \( C \) for a fixed input string \( \gamma \in \text{Rep}(G, a) \).
- Fix all input bits (as in \( \gamma \)) that do not correspond to potential edges between the spheres \( S_i \) and \( S_{i+1} \), for \( i < m \).
- For all \( i < m \) and all \( u \in S_i(a), v \in S_{i+1}(a) \) consider the potential edges \( e = \{u, v\}, e' = \{\pi(u), \pi(v)\}, \tilde{e} = \{u, \pi(v)\}, \tilde{e}' = \{\pi(u), v\} \).
- Replace input gates of \( C \) as follows:
  - \( e \) by \( (e \land \neg w_i) \)
  - \( e' \) by \( (e' \land \neg w_i) \)
  - \( \tilde{e} \) by \( (e \land w_i) \)
  - \( \tilde{e}' \) by \( (e' \land w_i) \)

- This yields a circuit \( \tilde{C} \) of the same size and depth as \( C \) which, on input \( w \in \{0, 1\}^m \) does the same as \( C \) on input \( (G_w, a) \). Thus, \( \tilde{C} \) accepts iff \( |w|_1 \) is even.
Proof of locality theorem — upper bound (5/5)

Key Lemma:
Let \( m \in \mathbb{N}, G = (V, E), a, b \in V \) such that \( \mathcal{N}_m^G(a) \approx \mathcal{N}_m^G(b) \) and \( \text{dist}(a, b) > 2m \).
Let circuit \( C \) accept all strings in \( \text{Rep}(G, a) \) and reject all strings in \( \text{Rep}(G, b) \).
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- Replace input gates of \( C \) as follows:
  \[
  e \text{ by } (e \land \neg w_i), \quad e' \text{ by } (e' \land \neg w_i), \\
  \tilde{e} \text{ by } (e \land w_i), \quad \tilde{e}' \text{ by } (e' \land w_i).
  \]
- This yields a circuit \( \tilde{C} \) of the same size and depth as \( C \) which, on input \( w \in \{0, 1\}^m \) does the same as \( C \) on input \( (G_w, a) \).
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- Consider \( C \) for a fixed input string \( \gamma \in \text{Rep}(G, a) \).
- Fix all input bits (as in \( \gamma \)) that do *not* correspond to potential edges between the spheres \( S_i \) and \( S_{i+1} \), for \( i < m \).
- For all \( i < m \) and all \( u \in S_i(a) \), \( v \in S_{i+1}(a) \) consider the potential edges \( e = \{u, v\} \), \( e' = \{\pi(u), \pi(v)\} \), \( \tilde{e} = \{u, \pi(v)\} \), \( \tilde{e}' = \{\pi(u), v\} \).
- Replace input gates of \( C \) as follows:
  - \( e \) by \( (e \wedge \neg w_i) \)
  - \( e' \) by \( (e' \wedge \neg w_i) \)
  - \( \tilde{e} \) by \( (e \wedge w_i) \)
  - \( \tilde{e}' \) by \( (e' \wedge w_i) \)

  This yields a circuit \( \tilde{C} \) of the same size and depth as \( C \) which, on input \( w \in \{0, 1\}^m \) does the same as \( C \) on input \( (G_w, a) \). Thus, \( \tilde{C} \) accepts iff \( |w|_1 \) is even.
Proof of locality theorem — upper bound (5/5)

Key Lemma:
Let $m \in \mathbb{N}$, $G = (V, E)$, $a, b \in V$ such that $\mathcal{N}_m^G(a) \cong \mathcal{N}_m^G(b)$ and $\text{dist}(a, b) > 2m$. Let circuit $C$ accept all strings in $\text{Rep}(G, a)$ and reject all strings in $\text{Rep}(G, b)$. Then there is a circuit $\tilde{C}$ of the same size & depth as $C$ computing parity on $m$ bits.

How to obtain $\tilde{C}$ from $C$?

- Consider $C$ for a fixed input string $\gamma \in \text{Rep}(G, a)$.
- Fix all input bits (as in $\gamma$) that do not correspond to potential edges between the spheres $S_i$ and $S_{i+1}$, for $i < m$.
- For all $i < m$ and all $u \in S_i(a)$, $v \in S_{i+1}(a)$ consider the potential edges $e = \{u, v\}$, $e' = \{\pi(u), \pi(v)\}$, $\tilde{e} = \{u, \pi(v)\}$, $\tilde{e}' = \{\pi(u), v\}$.
- Replace input gates of $C$ as follows:
  
  $e$ by $(e \land \neg w_i)$  
  $e'$ by $(e' \land \neg w_i)$  
  $\tilde{e}$ by $(e \land w_i)$  
  $\tilde{e}'$ by $(e' \land w_i)$

- This yields a circuit $\tilde{C}$ of the same size and depth as $C$ which, on input $w \in \{0, 1\}^m$ does the same as $C$ on input $(G_w, a)$.

Thus, $\tilde{C}$ accepts iff $|w|_1$ is even.
Theorem: (Anderson, Melkebeek, S., Segoufin ’11)

(a) For every query \(q\) expressible by \(Arb\)-invariant \(FO\) there is a \(c \in \mathbb{N}\) such that \(q\) is \((\log n)^c\)-local.

(b) For every \(d \in \mathbb{N}\) there is a \((+, \times)\)-invariant \(FO\) query that is not \((\log n)^d\)-local.

The query \(q_d(x)\) states:

1. The graph has at most \((\log n)^{d+1}\) non-isolated vertices.
   
   (Use the polylog-counting capability of \(FO(+, \times)\))

2. Node \(x\) is reachable from a node that belongs to a triangle.
   
   (Show that in graphs satisfying (1), reachability by paths of length \((\log n)^{d+1}\) can be expressed in \((+, \times)\)-invariant \(FO\))

Note: This query is not \((\log n)^d\)-local.
**Proof of locality theorem — lower bound (1/2)**

**Theorem:** *(Anderson, Melkebeek, S., Segoufin ’11)*

(a) For every query $q$ expressible by Arb-invariant FO there is a $c \in \mathbb{N}$ such that $q$ is $(\log n)^c$-local.

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2. Node $x$ is reachable from a node that belongs to a triangle.  
   (Show that in graphs satisfying (1), reachability by paths of length $(\log n)^{d+1}$ can be expressed in $(+, \times)$-invariant FO)

Note: This query is not $(\log n)^d$-local.
Proof of locality theorem — lower bound (1/2)

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(2) Node $x$ is reachable from a node that belongs to a triangle.

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Note: This query is not $(\log n)^d$-local.
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Proof of locality theorem — lower bound (1/2)

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1. The graph has at most \((\log n)^{d+1}\) non-isolated vertices.

   (Use the polylog-counting capability of \( \text{FO}(+ , \times) \))

2. Node \( x \) is reachable from a node that belongs to a triangle.

   (Show that in graphs satisfying (1), reachability by paths of length \((\log n)^{d+1}\) can be expressed in \((+ , \times)\)-invariant FO)

Note: This query is not \((\log n)^d\)-local.
Proof of locality theorem — lower bound (2/2)

**Goal:** Show that in graphs with \( \leq (\log n)^c \) non-isolated vertices, reachability by paths of length \( (\log n)^c \) can be expressed in \((+\times)\)-invariant FO.

**Lemma:** (Durand, Lautemann, More ’07)

For every \( c \in \mathbb{N} \) there is a \( \text{FO}(<, +, \times, S) \)-formula \( \text{bij}_c(x, y) \) such that for all \( n \in \mathbb{N} \), all \( S \subseteq [n] := \{0, \ldots, n-1\} \), all \( a, i < n \) we have

\[
([n], <, +, \times, S) \models \text{bij}_c(a, i) \iff |S| < (\log n)^c \quad \text{and} \quad a \text{ is the } i\text{-th smallest element of } S.
\]

- Using this, identify the non-isolated vertices with numbers \( \leq (\log n)^c \) and represent them by bitstrings of length \( c \log \log n \).
- Identify an arbitrary vertex of \( G \) with a number \( < n \), whose binary representation encodes a sequence of \( \ell(n) := \frac{\log n}{c \log \log n} \) non-isolated vertices.
- Use this to express that there is a path of length \( \ell(n) \) from node \( x \) to node \( y \).
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Proof of locality theorem — lower bound (2/2)

**Goal:** Show that in graphs with \( \leq (\log n)^c \) non-isolated vertices, reachability by paths of length \( (\log n)^c \) can be expressed in \((+, \times)\)-invariant FO.

**Lemma:** *(Durand, Lautemann, More ’07)*

For every \( c \in \mathbb{N} \) there is a FO\((<, +, \times, S)\)-formula \( \text{bij}_c(x, y) \) such that for all \( n \in \mathbb{N}, all \ S \subseteq [n] := \{0, \ldots, n-1\} \), all \( a, i < n \) we have

\[
([n], <, +, \times, S) \models \text{bij}_c(a, i) \iff |S| < (\log n)^c \quad \text{and} \quad a \text{ is the } i\text{-th smallest element of } S.
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- Using this, identify the non-isolated vertices with numbers \( < (\log n)^c \) and represent them by bitstrings of length \( c \log \log \log n \).
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Introduction

Invariant Logics

Non-expressibility Results for Arb-invariant FO

Final Remarks

Proof of locality theorem — lower bound (2/2)

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Overview

Introduction

Invariant logics

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Final Remarks
Gaifman locality

If $\mathcal{N}_r^G(a) \cong \mathcal{N}_r^G(b)$ then $(a \in q(G) \iff b \in q(G))$.

*Known:*

- Queries definable in order-invariant FO are Gaifman-local with respect to a constant locality radius. (Grohe, Schwentick ’98)
- Queries definable in Arb-invariant FO are Gaifman-local with respect to a poly-logarithmic locality radius. (Anderson, Melkebeek, S., Segoufin ’11)

*Open Question:*

- How about addition-invariant FO — is it Gaifman-local with respect to a constant locality radius?
Hanf locality

A graph property $\mathcal{P}$ is Hanf-local w.r.t. locality radius $r$, if any two graphs having the same $r$-neighbourhood types with the same multiplicities, are not distinguished by $\mathcal{P}$.

Known:

- Properties of strings or trees definable by order-invariant FO are Hanf-local w.r.t. a constant locality radius. (Benedikt, Segoufin '09)
- Properties of strings definable by Arb-invariant FO are Hanf-local w.r.t. a poly-logarithmic locality radius. (Anderson, Melkebeek, S., Segoufin '11)

Open Question:

- Do these results generalise from strings to arbitrary finite graphs?
Decidable Characterisations

Open Question:
Are there decidable characterisations of

- order-invariant FO?
- addition-invariant FO?
- \((+,\times)\)-invariant FO?

Known:

- On finite strings and trees: order-invariant FO \(\equiv\) FO. (Benedikt, Segoufin '10)
- On finite coloured sets: addition-invariant FO \(\equiv\) FO enriched by “cardinality modulo” quantifiers. (S., Segoufin '10)
Thank You!