Pseudo-analytic structures: model theory and algebraic geometry

B. Zilber

University of Oxford

Manchester 2012
Strongly minimal structures. Examples

(1) **Trivial**

$G$-set $M = \{M, g \cdot \} g \in G$, for $G$ a group acting on $M$ "in a nice way". E.g. the upper half plane $\mathcal{H}$ with the action of $GL^+(2, \mathbb{Q})$.

(2) **Linear**

Abelian divisible torsion-free groups; Abelian groups of prime exponent; Vector spaces over a given division ring $K$.

(3) **Algebraically closed fields** in the language $(+, \cdot, =)$
Dimension notions

for finite $X \subset M$:

(1) Trivial structures: the number of "generic" $G$-orbits in $G \cdot X$
(2) Linear structures: the linear dimension $\text{lin.d}_Q(X)$ of $\langle X \rangle$
(3) Algebraically closed fields: the transcendence degree $\text{tr.d}(X)$ over the prime subfield.

Dual notion: the dimension of an algebraic variety $V$ over $F$

$$\dim V = \max\{ \text{tr.d}_F(x_1, \ldots, x_n) \mid (x_1, \ldots, x_n) \in V \}.$$
Three basic geometries of stability theory

(1) **Trivial geometry**
(2) **Linear geometry**
(3) **Algebraic geometry.**

**Trichotomy Conjecture**

*Any uncountably categorical structure is reducible to (1) - (3)?*
Hrushovski’s construction of new stable structures

Given a class of structures $\mathbf{M}$ with a dimension notions $d_1$, and $d_0$ we want to consider a new function $f$ on $\mathbf{M}$ and extend the dimension theory.

On $(\mathbf{M}, f)$ introduce a predimension

$$\delta(x_1, \ldots, x_n) = d_1(x_1, \ldots, x_n, f(x_1), \ldots, f(x_n)) - d_0(x_1, \ldots, x_n).$$

We must assume

$$\delta(X) \geq 0, \text{ for all finite } X \subset M$$

(Hrushovski inequality).

Use the Fraisse amalgamation procedure in the class $(\mathbf{M}, f)$ respecting the predimension $\delta$.

Under certain tameness assumptions on $\mathbf{M}$, $d_1$ and $d_0$ this gives rise to a complete theory of generic structure, which is stable and even strongly minimal with a geometry distinct from (1)-(3).
Variations (two-sorted fusion)

\[(M_1; L_1) \quad \downarrow f \quad (M_2; L_2)\]

\[\delta(X) = d_1(X) + d_2(f(X)) - d_0(X)\]

\(d_1 = \text{dimension in } M_1\), \(d_2 = \text{dimension in } M_2\), \(d_0 = \text{dimension for the } f\text{-invariant part of both structures.}\)
Example. (Hrushovski, 1992)

\[(F_1; +, \cdot) \quad \downarrow f \quad (F_2; +, \cdot)\]

\[d_1(X) = \text{tr}.d F_1(X), \quad d_2(Y) = \text{tr}.d F_2(Y), \quad d_0(X) = |X|, \quad f \text{ bijection}\]

Can be seen as a **fusion** of two pregeometries with dimensions \(d_1\) and \(d_2\), preserving a common part corresponding to predimension \(d_0\).
Are Hrushovski structures mathematical pathologies?

Observation: If $M$ is a field of characteristic 0 and we want $f = \exp$ to be a group homomorphism:

$$\exp(x_1 + x_2) = \exp(x_1) \cdot \exp(x_2),$$

then the corresponding predimension must be

$$\delta_{\exp}(X) = \text{tr.d}(X \cup \exp(X)) - \text{lin.d}_Q(X) \geq 0.$$

The Hrushovski inequality, in the case of the complex numbers and $\exp = \exp$, is equivalent to

$$\text{tr.d}(x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n}) \geq n,$$

assuming that $x_1, \ldots, x_n$ are linearly independent. This is the Schanuel conjecture.
Can we carry out Hrushovski construction for $\delta_{\text{exp}}$?

Issues:

(i) not enough tameness in $\delta_{\text{exp}}$

(ii) the natural prototype $C_{\text{exp}}$ has the ring $\mathbb{Z}$ as a definable substructure.

Solution. Treat this case in a non-elementary setting.

**Theorem** (2003) The amended Hrushovski construction for fields with pseudo-exponentiation produces an $L_{\omega_1,\omega}(Q)$-theory $T_{\text{exp}}$ of a field with pseudo-exponentiation, categorical in all uncountable powers.

$Q$ is a quantifier "there exists uncountably many".
Axioms of $T_{\text{exp}}$

The language $(+, \cdot, \text{ex}, 0, 1)$

$\text{ACF}_0$ algebraically closed fields of characteristic 0;

$\text{EXP1: } \text{ex}(x_1 + x_2) = \text{ex}(x_1) \cdot \text{ex}(x_2)$;

$\text{EXP2: } \ker \text{ex} = \pi \mathbb{Z}$;

$\text{SCH: }$ for any finite $X$

$$\delta(X) = \text{tr.d}(X \cup \text{ex}(X)) - \text{lin.d}_Q(X) \geq 0$$

this is $L_{\omega_1, \omega}$.
Axioms of $T_{\text{exp}}$, continued

As a result of Fraïssé amalgamation models of $T_{\text{exp}}$ are existentially closed with respect to embedding respecting $\delta_{\text{exp}}$.

EC: For any rotund system of polynomial equations

$$P(x_1, \ldots, x_n, y_1, \ldots, y_n) = 0$$

there exists a (generic) solution satisfying

$$y_i = \text{ex}(x_i) \quad i = 1, \ldots, n.$$ 

(this is basically first order, but "generic" requires $L_{\omega_1,\omega}$.)

And

**Countable closure property**

CC: For maximal rotund systems of equations the set of solutions is at most countable. $L_{\omega_1,\omega}(Q)$
Reformulation

**Theorem** Given an uncountable cardinal $\lambda$, there is a unique model of axioms $\text{ACF}_0 + \text{EXP} + \text{SCH} + \text{EC} + \text{CC}$ of cardinality $\lambda$.

This is a consequence of Theorems A and B:

**Theorem A** The $L_{\omega_1,\omega}(Q)$-sentence

$$\text{ACF}_0 + \text{EXP} + \text{SCH} + \text{EC} + \text{CC}$$

is axiomatising a *quasi-minimal excellent abstract elementary class* ($\text{AEC}$).

**Theorem B** (Essentially S. Shelah 1983, see also J. Baldwin 2010) A quasi-minimal excellent AEC has a unique model in any uncountable cardinality.

**Remark.** "Excellence" is essential. The earlier Kiesler’s theory of homogeneous $L_{\omega_1,\omega}$-categoricity is not applicable here.
Theorem A

The proof reduces to the following arithmetic-algebraic facts:

(i) the action of $\text{Gal}(\bar{\mathbb{Q}} : \mathbb{Q})$ on $\text{Tors}$ is maximal possible (Dedekind);

(ii) given $k$, a finitely generated extension of (1) $\mathbb{Q}(\text{Tors})$ or of (2) $\bar{\mathbb{Q}}$, the algebraic closure of $\mathbb{Q}$, and given a finitely generated subgroup $A$ of the multiplicative group $G_m(k) := k^\times$, the group

$$\text{Hull}_k(A) / T(k) \cap \text{Hull}_k(A),$$

where $T(k) = \begin{cases} \text{Tors}, & \text{if (1)} \\ \bar{\mathbb{Q}}, & \text{if (2)} \end{cases}$

is free. (Follows from Kummer theory)

(iii) similar to (ii) but for $k =$ composite of finite independent system of algebraically closed fields (Bays and Z.)

In fact, (i)-(iii) is equivalent to categoricity of $T_{\text{exp}}$ modulo model theory.
Theorem A for other transcendental functions

We need to know a "complete system of functional equations" and the "Schanuel conjecture" for the function(s).

The Weierstrass function \( \wp(\tau, z) \) (as a function of \( z \)) and the structure on the elliptic curve \( E_{j(\tau)} \):

\[
\langle \wp(\tau, z), \wp(\tau, z)' \rangle : \mathbb{C} \rightarrow E_{j(\tau)} \setminus \{\infty\} \subset \mathbb{C}^2
\]

Since

\[
(\wp')^2 = 4\wp^3 - g_2 \cdot \wp - g_3(\tau), \quad g_2 = g_2(\tau), \quad g_3 = g_3(\tau),
\]

the problem reduces to the structure

\[
(\mathbb{C}, +, \cdot, \wp(z))
\]

The Schanuel-type conjecture was deduced from the André conjecture on 1-motives by C.Bertolin.
Theorem A for $\mathfrak{P}(\tau, z)$

M.Gavrilovich, M.Bays, J.Kirby, B.Hardt (published and work in progress):

(i) the action of $\text{Gal}(\tilde{\mathbb{Q}}(j(\tau)) : \mathbb{Q}(j(\tau)))$ on $E_{j(\tau)}(\text{Tors})$ is maximal possible (essentially, the hard theorem of J.-P. Serre);

(ii) given $k$, a finitely generated extension of (1) $\mathbb{Q}(E_{j(\tau)}(\text{Tors}))$ or of (2) $\tilde{\mathbb{Q}}(j(\tau))$, the algebraic closure of $\mathbb{Q}(j(\tau))$, and given a finitely generated subgroup $A$ of the group $E_{j(\tau)}(k)$, the group

$$\text{Hull}_k(A)/T(k) \cap \text{Hull}_k(A), \text{ where } T(k) = \begin{cases} E_{j(\tau)}(\text{Tors}) & \text{if (1)} \\ \tilde{\mathbb{Q}}(j(\tau)) & \text{if (2)} \end{cases}$$

is free. (Mordell-Weil, Ribet)

(iii) follows from (ii) in general for commutative algebraic groups (Bays–Hardt, using Shelah’s techniques)
Theorem A for other transcendental functions

Weierstrass function \( \wp(\tau, z) \) as function of \( \tau \) and \( z \) still poorly understood, even at the level of functional equations and Schanuel-type conjecture.

Work on function \( j(\tau) \) (modular invariant) in progress, A.Harris:
(i) adelic Mumford-Tate conjecture for Abelian varieties = product of elliptic curves. Theorem of Serre.
(ii) Shimura reciprocity and other elements of the theory of \( j \)-invariant.
(iii) Bays-Hardt as above.

Further transcendental functions are of interest. First of all the uniformising functions for (mixed) Shimura varieties (includes semi-abelian varieties).
Is $T_{\text{exp}}$ the actual theory of exp?

**Conjecture.** $\mathbb{C}_{\text{exp}}$ is the unique model of $T_{\text{exp}}$ of cardinality continuum.

This is equivalent to

**Conjecture.** $\mathbb{C}_{\text{exp}}$ satisfies SCH and EC.

Work on comparative analysis of properties of $\mathbb{C}_{\text{exp}}$ and $T_{\text{exp}}$.

A. Macintyre, A. Wilkie, D. Marker, P. D’Aquino, G. Terzo, A. Shkop, V. Mantova, B. Z. and others.

**Conclusion so far.** Hrushovski’s construction is behind classical analytic-algebraic geometry.
First order framework

Recall the issues with the first order treatment:

(i) not enough tameness in $\delta_{\exp}$

(ii) the natural prototype $C_{\exp}$ has the ring $\mathbb{Z}$ as a definable substructure.

Solution for (ii): Work out first order axioms for the pseudo-exponentiation \textit{modulo the complete arithmetic}.

The analysis of (i) lead to the possible remedy

\textbf{Conjecture on Intersection with Tori (CIT), 2001.}
(Formulation in model-theoretic form, using a 2-sorted predimension)

Let $^*\mathbb{C}$ and $^*\mathbb{Q}$ be nonstandard models of complex and rational number fields. Then for any finite $X \subset ^*\mathbb{C}$,

$$
\delta(X) := \text{tr.d}(\exp(X)/\mathbb{C}) + \text{lin.d.}^*\mathbb{Q}(X/\mathbb{C}) - \text{lin.d}_\mathbb{Q}(X/\ker) \geq 0.
$$
First order framework

Recall \( T_{\exp} : \text{ACF}_0 + \text{EXP} + \text{SCH} + \text{EC} + \text{CC} \)

**Theorem** (Kirby, Z., 2011) The axiom SCH (Schanuel condition) is first order axiomatisable (over the kernel) iff CIT is true.
In this case the complete system of first order axioms of pseudo-exponentiation can be written down explicitly modulo the complete arithmetic.

In effect, one can say that the models of the first order theory "split" into two mutually "orthogonal" components: the kernel (arithmetic) and an \( \omega \)-stable part.

The theory is \( \omega \)-stable over the arithmetic.
CIT and Pink’s conjecture

CIT can be reformulated in an equivalent algebro-geometric form. Also, in the form applicable to semi-abelian varieties and indeed to any context where Schanuel-type conjecture makes sense.

Proposition. CIT implies Mordell-Lang (and Manin-Mumford) conjectures.

Later an equivalent of CIT was formulated by Bombieri, Masser and Zanier.

General form of CIT for mixed Shimura varieties was formulated by R.Pink in 2005. This includes André-Oort conjecture about special points on Shimura varieties. This is now referred to as Z.-Pink conjecture.
J. Pila’s idea of extending the Bombieri-Pila method of counting rational points on transcendental ovals to o-minimal context.

Pila-Wilkie’s theorem (2005) establishes an upper bound for the number of rational points on the transcendental part of sets definable in o-minimal expansions of the reals.

Pila and Zannier (2009) showed how to solve Manin-Mumford-type problem using Pila-Wilkie and weak CIT (Ax’s theorem, "Ax-Schanuel").

This method developed into the solution of a number of cases of André-Oort and ZP.