Patterns with Bounded Treewidth

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Pattern languages

\[ x_1 \text{ } aa \text{ } x_2 \text{ } x_1 \text{ } x_2 \text{ } cb \text{ } x_1 \]
Pattern languages

\[ x_1 \text{ aa } x_2 \text{ x}_1 \text{ x}_2 \text{ cb } x_1 \]

\[ \text{acaaabcbacaacabcbacbac} \]
Pattern languages

\[
x_1 \textcolor{green}{aa} x_2 x_1 x_2 \textcolor{green}{cb} x_1
\]

\[
\textcolor{green}{aca}aabcbaacabcbacbcba\textcolor{green}{c}bc
\]
Pattern languages

\[ x_1 \text{ aa } x_2 \text{ x_1 } x_2 \text{ cb } x_1 \]

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Pattern languages

\[ x_1 \text{ aa } x_2 \text{ } x_1 \text{ } x_2 \text{ cb } x_1 \]

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Pattern languages

\[ x_1 \ aa \ x_2 \ x_1 \ x_2 \ cb \ x_1 \]

\[ acaaaabcbaacabcbaacbacbac \]

\[ \{ w \mid w = u \ aa \ v \ u \ v \ cb \ u, \ where \ u, \ v \in \{a, b, c\}^* \}. \]
The membership problem for pattern languages is NP-complete (Angluin, 80).
The membership problem for pattern languages

**Theorem**

The membership problem for pattern languages is NP-complete (Angluin, 80).

Main research task: Find classes of pattern languages with a polynomial membership problem.
Previous results

- Brute-force algorithm with runtime $O(|w|^k)$, where $k$ is the number of variables.
- A pattern language is a regular language if
  - it is unary or
  - no variable in the pattern is repeated.
- $|w| \leq k$ for a constant $k$ (Geilke, Zilles, ALT 2011)
- *Non-cross* patterns (e.g., $x_1x_1x_1x_2x_2x_3x_3x_3$) (Shinohara, 82).
- Patterns with bounded *variable distance* (R., S., CIAA 2010).
**Definition (Relational structure)**

A relational vocabulary $\tau$ is a finite set of relation symbols. Every relation symbol $R \in \tau$ has an arity $\text{ar}(R) \geq 1$. A $\tau$-structure $\mathcal{A}$ comprises a finite set $A$ called the universe and, for every $R \in \tau$, an interpretation $R^\mathcal{A} \subseteq A^{\text{ar}(R)}$.

**Example**

Every graph can be interpreted as a relational structure $\mathcal{G}$, where the universe $V$ is the set of vertices and the edges are given as an interpretation of a binary relation $E$.

We only consider relations with arity of at most 2.
**Homomorphism problem for relational structures**

**Definition (Homomorphism)**

Let $\mathcal{A}$ and $\mathcal{B}$ be $\tau$-structures with universes $A$ and $B$, respectively. A *homomorphism* from $\mathcal{A}$ to $\mathcal{B}$ is a mapping $h : A \rightarrow B$ such that for all $R \in \tau$ and for all $a_1, a_2, \ldots, a_{\ar(R)} \in A$, $(a_1, a_2, \ldots, a_{\ar(R)}) \in R^A$ implies $(h(a_1), h(a_2), \ldots, h(a_{\ar(R)})) \in R^B$.

**Definition (Homomorphism problem)**

The *homomorphism problem* $HOM$ is the problem of deciding, for any structures $\mathcal{A}$ and $\mathcal{B}$, whether there exists a homomorphism from $\mathcal{A}$ to $\mathcal{B}$. For any set of structures $C$, by $HOM(C)$ we denote the homomorphism problem, where the left hand structure is restricted to be from $C$. 

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Homomorphism problem for relational structures

Theorem

The homomorphism problem for relational structures is NP-complete.
\( \alpha \)-structures

\[
\alpha := x_1 \ x_2 \ x_1 \ x_3 \ x_2 \ x_3 \ b \ x_1 \ x_2 \ x_1
\]
α-structures

\[ \alpha := x_1 \quad x_2 \quad x_1 \quad x_3 \quad x_2 \quad x_3 \quad b \quad x_1 \quad x_2 \quad x_1 \]
\( \alpha \)-structures

\[
\alpha := x_1 \; x_2 \; x_1 \; x_3 \; x_2 \; x_3 \; b \; x_1 \; x_2 \; x_1
\]
\[ \alpha := x_1 \ x_2 \ x_1 \ x_3 \ x_2 \ x_3 \ b \ x_1 \ x_2 \ x_1 \]

\[ 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 7 \rightarrow 8 \rightarrow 9 \rightarrow 10 \]
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α-structures

α := x₁ x₆ x₃ x₄ x₅ x₆ b x₁ x₂ x₁
\( \alpha \)-structures

\[ \alpha := x_1 \ x_2 \ x_1 \ x_3 \ x_2 \ x_3 \ b \ x_1 \ x_2 \ x_1 \]
$\alpha$-structures

$\alpha := x_1 \ x_2 \ x_1 \ x_3 \ x_2 \ x_3 \ b \ x_1 \ x_2 \ x_1$
\( \alpha \text{-structures} \)

\[ \alpha := x_1 \ x_2 \ x_2 \ x_1 \ x_3 \ x_2 \ x_3 \ b \ x_1 \ x_2 \ x_1 \]
\( \alpha \)-structures

\[
\alpha := x_1 \ x_2 \ x_1 \ x_3 \ x_2 \ x_3 \ b \ x_1 \ x_2 \ x_1
\]
We denote $\alpha$-structures by $\mathcal{A}_\alpha$ and the standard $\alpha$-structure by $\mathcal{A}_\alpha^S$. 

$\alpha := x_1 \ x_2 \ x_1 \ x_3 \ x_2 \ x_3 \ b \ x_1 \ x_2 \ x_1$
$w$-structures

\[ w := abab \]
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1, 2, 3, 4

1, 2, 2, 3, 3, 4

2, 4, 1, 4, 1, 3
$w$-structures

\[ w := abab \]

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$w$-structures

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$w := abab$

We denote $w$-structures by $A_w$. 

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$w$-structures

$w := abab$

We denote $w$-structures by $A_w$. 
Membership problem \(\leq\) homomorphism problem

**Lemma**

A word \(w\) is in the pattern language of \(\alpha\) if and only if there exists a homomorphism from \(A_\alpha\) to \(A_w\), where \(A_\alpha\) is some \(\alpha\)-structure.
A meta theorem

Theorem (Freuder, 90)

Let $C$ be a class of relational structures with bounded treewidth. Then $HOM(C)$ can be solved in polynomial time.
### Theorem (Freuder, 90)

Let $C$ be a class of relational structures with bounded treewidth. Then $\text{HOM}(C)$ can be solved in polynomial time.

### Theorem (Meta theorem)

Let $P$ be a class of patterns and let $f$ be a polynomial time computable function that maps every $\alpha \in P$ to an $\alpha$-structure. If, for some constant $k$, $\max \{ \text{tw}(f(\alpha)) \mid \alpha \in P \} \leq k$, then the membership problem for $P$ is decidable in polynomial time.
A meta theorem

Theorem (Freuder, 90)

Let $C$ be a class of relational structures with bounded treewidth. Then $HOM(C)$ can be solved in polynomial time.

Theorem (Meta theorem)

Let $P$ be a class of patterns and let $f$ be a polynomial time computable function that maps every $\alpha \in P$ to an $\alpha$-structure. If, for some constant $k$, $\max\{\text{tw}(f(\alpha)) \mid \alpha \in P\} \leq k$, then the membership problem for $P$ is decidable in polynomial time.

Research task: Find classes of patterns with bounded treewidth.
Scope coincidence degree

Let $\alpha$ be a pattern.

- For every $y \in \text{var}(\alpha)$, the scope of $y$ in $\alpha$ is defined by $\text{sc}_\alpha(y) := \{i, i + 1, \ldots, j\}$, where $i$ is the leftmost and $j$ the rightmost position of $y$ in $\alpha$.
- The scopes of $y_1, y_2, \ldots, y_k \in \text{var}(\alpha)$ coincide in $\alpha$ if and only if $\bigcap_{1 \leq i \leq k} \text{sc}_\alpha(y_i) \neq \emptyset$.
- The scope coincidence degree of $\alpha$ (scd($\alpha$)) is the maximum number of variables in $\alpha$ such that their scopes coincide.
Let $\alpha$ be a pattern.

- For every $y \in \text{var}(\alpha)$, the *scope of $y$ in $\alpha$* is defined by $\text{sc}_\alpha(y) := \{i, i+1, \ldots, j\}$, where $i$ is the leftmost and $j$ the rightmost position of $y$ in $\alpha$.
- The scopes of $y_1, y_2, \ldots, y_k \in \text{var}(\alpha)$ *coincide in $\alpha$* if and only if $\bigcap_{1 \leq i \leq k} \text{sc}_\alpha(y_i) \neq \emptyset$.
- The *scope coincidence degree of $\alpha$* ($\text{scd}(\alpha)$) is the maximum number of variables in $\alpha$ such that their scopes coincide.
Let $\alpha$ be a pattern.

- For every $y \in \text{var}(\alpha)$, the *scope of $y$ in $\alpha$* is defined by $\text{sc}_\alpha(y) := \{i, i+1, \ldots, j\}$, where $i$ is the leftmost and $j$ the rightmost position of $y$ in $\alpha$.
- The scopes of $y_1, y_2, \ldots, y_k \in \text{var}(\alpha)$ *coincide in $\alpha$* if and only if $\bigcap_{1 \leq i \leq k} \text{sc}_\alpha(y_i) \neq \emptyset$.
- The *scope coincidence degree* of $\alpha$ ($\text{scd}(\alpha)$) is the maximum number of variables in $\alpha$ such that their scopes coincide.

\[ \alpha_1 := \begin{array}{cccccc}
 x_1 & x_2 & x_1 & x_3 & x_2 & x_3 & x_1 & x_2 & x_3 \\
\end{array} \quad \text{scd}(\alpha_1) = 3 \]

\[ \alpha_2 := \begin{array}{cccccc}
 x_1 & x_2 & x_1 & x_1 & x_2 & x_3 & x_2 & x_3 & x_3 \\
\end{array} \quad \text{scd}(\alpha_2) = 2 \]
Scope coincidence degree

Lemma

Let $\alpha$ be a pattern. Then $\text{tw}(A^s_\alpha) \leq \text{scd}(\alpha) + 1$. 

Theorem

Let $k \in \mathbb{N}$ and $P := \{\alpha \mid \text{scd}(\alpha) \leq k\}$. The membership problem for the class $P$ is decidable in polynomial time.
Mildly entwined patterns

Let $\alpha$ be a pattern.

- Two variables $x, y \in \text{var}(\alpha)$ are entwined iff
  \[ \alpha = \beta \cdot x \cdot \gamma_1 \cdot y \cdot \gamma_2 \cdot x \cdot \gamma_3 \cdot y \cdot \delta. \]
- If no two variables are entwined, then $\alpha$ is nested.
- $\alpha$ is closely entwined iff $\alpha = \beta \cdot x \cdot \gamma_1 \cdot y \cdot \gamma_2 \cdot x \cdot \gamma_3 \cdot y \cdot \delta$ with $|\gamma_2|_x = |\gamma_2|_y = 0$ implies $\gamma_2 = \varepsilon$.
- $\alpha$ is mildly entwined iff it is closely entwined and, for every $x \in \text{var}(\alpha)$, if $\alpha = \beta \cdot x \cdot \gamma \cdot x \cdot \delta$ with $|\gamma|_x = 0$, then $\gamma$ is nested.
Mildly entwined patterns

\[ \alpha = \ x_1 \ x_3 \ x_4 \ x_4 \ x_3 \ x_3 \ x_1 \ x_2 \ x_3 \ x_5 \ x_5 \ x_2 \ x_5 \ x_6 \ x_6 \ x_2 \]
Mildly entwined patterns

\[ \alpha = x_1 x_3 x_4 x_4 x_3 x_3 x_1 x_2 x_3 x_5 x_5 x_2 x_5 x_6 x_6 x_2 \]

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- for every \( x \in \text{var}(\alpha) \), if \( \alpha = \beta \cdot x \cdot \gamma \cdot x \cdot \delta \) with \( |\gamma|_x = 0 \), then \( \gamma \) is nested.
Mildly entwined patterns

\[ \alpha = x_1 x_3 x_4 x_4 x_3 x_3 x_1 x_2 x_3 x_5 x_5 x_2 x_5 x_6 x_6 x_2 \]

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Mildly entwined patterns

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Mildly entwined patterns

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- for every \( x \in \text{var}(\alpha) \), if \( \alpha = \beta \cdot x \cdot \gamma \cdot x \cdot \delta \) with \( |\gamma|_x = 0 \), then \( \gamma \) is nested.

\[ \Rightarrow \alpha \text{ is mildly entwined.} \]
Mildly entwined patterns

\[ \alpha = x_1 \ x_3 \ x_4 \ x_4 \ x_3 \ x_1 \ x_2 \ x_3 \ x_5 \ x_5 \ x_2 \ x_5 \ x_6 \ x_6 \ x_2 \]
Mildly entwined patterns

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\[\begin{array}{cccccccccccccccc}
1 & \rightarrow & 2 & \rightarrow & 3 & \rightarrow & 4 & \rightarrow & 5 & \rightarrow & 6 & \rightarrow & 7 & \rightarrow & 8 & \rightarrow & 9 & \rightarrow & 10 & \rightarrow & 11 & \rightarrow & 12 & \rightarrow & 13 & \rightarrow & 14 & \rightarrow & 15 & \rightarrow & 16
\end{array}\]
Mildly entwined patterns

\[ \alpha = x_1 \ x_3 \ x_4 \ x_4 \ x_3 \ x_1 \ x_2 \ x_3 \ x_5 \ x_5 \ x_2 \ x_5 \ x_6 \ x_6 \ x_2 \]

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Mildly entwined patterns

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Mildly entwined patterns

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**Definition**

A graph is *outerplanar* iff it can be drawn in a planar way such that no vertex is entirely surrounded by edges (or, equivalently, all vertices lie on the exterior face).
Mildly entwined patterns

**Lemma**

A pattern $\alpha$ is mildly entwined if and only if $A_{\alpha}^s$ is outerplanar.
Mildly entwined patterns

**Lemma**

A pattern $\alpha$ is mildly entwined if and only if $A^s_\alpha$ is outerplanar.

**Theorem (Bodlaender 86)**

If $G$ is an outerplanar graph, then $\text{tw}(G) \leq 2$. 
Mildly entwined patterns

**Lemma**

A pattern $\alpha$ is mildly entwined if and only if $A_{\alpha}^s$ is outerplanar.

**Theorem (Bodlaender 86)**

If $G$ is an outerplanar graph, then $\text{tw}(G) \leq 2$.

**Theorem**

Let $P := \{\alpha \mid \alpha$ is mildly entwined\}. The membership problem for the class $P$ is decidable in polynomial time.
Mildly entwined vs. bounded scope coincidence degree

Observation

For every $k$, $k \geq 2$, $\{\alpha \mid \text{scd}(\alpha) \leq k\} \neq \{\alpha \mid \alpha \text{ is mildly entwined}\}$. 

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