Tractable Sets of the Generalized Interval Algebra

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Abstract. To offer a generic frame which groups together several interval algebra generalizations, we simply define a generalized interval as a tuple of intervals. After introducing the generalized relations we focus on the consistency problem of generalized constraint networks and we present sets of generalized relations for which this problem is tractable, in particular the set of the strongly-preconvex relations.

1 Introduction

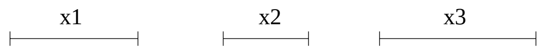
In many areas of Artificial Intelligence, Allen's Interval Algebra (IA) is used for temporal representation and reasoning [1]. Allen takes intervals as primitive temporal entities and considers 13 atomic relations between these intervals (fig. 1). IA was generalized in numerous ways and particularly at the level of the basic entities considered. Notably, many formalisms [11, 12, 6, 3, 5] consider tuples of intervals satisfying particular atomic relations of IA as basic entities instead of intervals. In the continuity of Balbiani *et al.* [2] who define a representation subsuming all these formalisms by introducing generalized intervals, we define a framework still more flexible. This representation enables us to reason with qualitative constraints on generalized intervals still more "general". In the following section we define the generalized intervals and the relations we consider between these entities. Then in section 3 we will remind the notion of convexity and we will introduce the weak preconvexity in section 4 – both notions introduced respectively by Nökel [9] and by Ligozat [7] for IA. Section 5 will be devoted to the generalized constraint networks and to the path-consistency and weak path-consistency methods. In sections 6 and 7 we will characterize sets of generalized relations for which the consistency problem of a generalized network is a polynomial problem. Finally we will show that these results of complexity subsume the ones obtained in [2].

2 The Generalized Interval Algebra

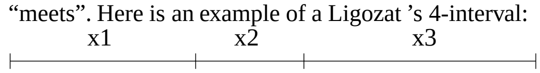
2.1 The Generalized Intervals

We define a generalized interval X to dimension p (with $p > 0$), called a p-interval, as a p-tuple of intervals $x_1, \ldots, x_p$ of the real line. This definition is very general and includes the definition of the generalized intervals of Ladkin [11, 12], Ligozat [6], and the one of the n-blocks [5], an extension to any dimension n of the Allen's intervals. In all these definitions a generalized interval is a tuple of intervals satisfying specific constraints expressible with Allen's relations (see fig. 1):

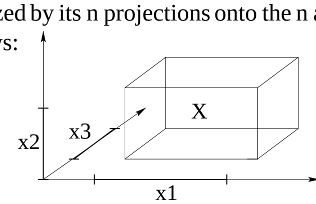
− for Ladkin a p-interval is a of p-tuple intervals $x_1, \ldots, x_p$ of the real line such that x_i and x_{i+1} satisfy the atomic relation "precedes". As an example of Ladkin's 3-interval we have:



− Ligozat defines a generalized p-interval as a tuple of $p-1$ intervals $x_1, \ldots, x_{p-1}$ such that x_i satisfies with x_{i+1} the atomic relation "meets". Here is an example of a Ligozat 's 4-interval:



− The n-blocks are the blocks of the Euclidean space of dimension n whose sides are parallel to the axe of some orthogonal base. A n-block is characterized by its n projections onto the n axe. An example of a 3-block follows:



Although the projections of a n-block are on distinct axe, we can represent them on a same line. Hence, a n-block can be represented by a n-tuple of intervals $x_1, \ldots, x_n$. Unlike the previous two cases, every pair (x_i, x_j) can satisfy any IA atomic relation.

So our notion of generalized interval is very general since it subsumes those previously cited. Moreover, we do not consider p-intervals with a fixed p. Two generalized intervals can have two different number of sub-intervals (contrary to the generalized intervals considered in [2]). Another difference with the generalized intervals of [2] is that our generalized intervals do not have the same rigid structure. In the next section we are going to present the relations considered between the generalized intervals.

2.2 The generalized relations

In the sequel we will denote the set of matrices $n \times p$ on a set E by $\mathcal{M}(E)_{n\times p}$ and the set of the 13 atomic relations of IA by $\mathcal{A}_{int}$.

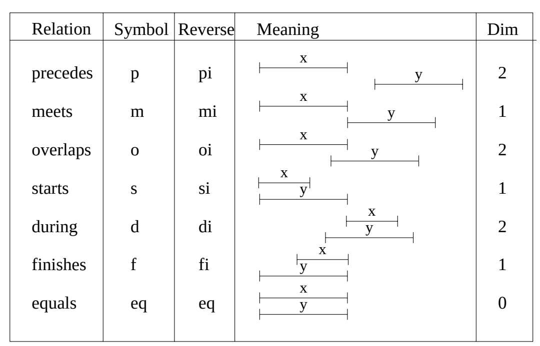


Figure 1. The atomic relations of IA: $\mathcal{A}_{int}$.

The set of the atomic relations between a p-interval and a q-interval,

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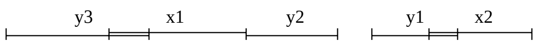
$\mathcal{A}_{p,q}$, is defined by the set of the matrices $p \times q$ of the IA atomic relations:

$$\mathcal{A}_{p,q} = \{A : A \in \mathcal{M}(\mathcal{A}_{int})_{p\times q}\}.$$

Let X be a p-interval, Y a q-interval and $A \in \mathcal{A}_{p,q}$, X and Y satisfy A, denoted by $X\ A\ Y$, iff $\forall i \in 1,\ldots,p$ and $\forall j \in 1,\ldots,q$, $x_i\ A_{ij}\ y_j$. A p-interval and a q-interval satisfy one, and only one, atomic relation from $\mathcal{A}_{p,q}$. These ones are complete and mutually exclusive. As an illustration, in the following figure are represented a 2-interval $X = (x_1, x_2)$ and a 3-interval $Y = (y_1, y_2, y_3)$:



We have $X \begin{pmatrix} b & m & oi \\ oi & bi & bi \end{pmatrix} Y$, $X \begin{pmatrix} eq & b \\ bi & eq \end{pmatrix} X$, $Y \begin{pmatrix} eq & bi & bi \\ b & eq & bi \\ b & b & eq \end{pmatrix} Y$.

Let us remark that there exist generalized atomic relations which can be never satisfy. For instance no pair of 2-intervals can satisfy the atomic relation $\begin{pmatrix} b & bi \\ bi & b \end{pmatrix}$. The set of relation considered between a p-interval and a q-interval in the generalized interval algebra is the power set of $\mathcal{A}_{p,q}$, i.e. $2^{\mathcal{A}_{p,q}}$. Given a relation $R \in 2^{\mathcal{A}_{p,q}}$, a p-interval X and a q-interval Y, X and Y satisfy R iff there exists $A \in R$ such that $X\ A\ Y$. Let a matrix $M \in \mathcal{M}(2^{\mathcal{A}_{int}})_{p\times q}$ be given, we will note $\prod M$ the relation of $2^{\mathcal{A}_{p,q}}$ defined by:

$$\prod M = \{A \in \mathcal{A}_{p,q} : A_{ij} \in M_{ij}, 1 \leq i \leq p \text{ and } 1 \leq j \leq q\}.$$

Let a relation $R \in 2^{\mathcal{A}_{p,q}}$ be, $R_{\downarrow ij}$, with $1 \leq i \leq p$ and $1 \leq j \leq q$, denotes the AI relation: $R_{\downarrow ij} = \{A \in \mathcal{A}_{int} : \exists B \in R, B_{ij} = A\}$, and $R_\downarrow$ is the matrix $\mathcal{M}(2^{\mathcal{A}_{int}})_{p\times q}$ defined by $(R_\downarrow)_{ij} = R_{\downarrow ij}$. We can easily prove the following proposition:

Proposition 1 *Let $R, S \in 2^{\mathcal{A}_{p,q}}$ be. (a) $R \subseteq \prod R_\downarrow$, and (b) if $R \subseteq S$ then $R_{\downarrow ij} \subseteq S_{\downarrow ij}$, with $1 \leq i \leq p$ and $1 \leq j \leq q$.*

When $R = \prod R_\downarrow$ is satisfied we will say that R is saturated.

2.3 The Fundamental Operations

The fundamental operations, intersection ($\cap$), union ($\cup$), composition ($\circ$) and inverse ($^{-1}$) are defined on the set of the generalized relations. The binary operations intersection and union are the usual homonymous set operations. The operations inverse and composition are defined from the ones of IA:

Definition 1

- *Let $A \in \mathcal{A}_{p,q}$ be, $A^{-1} = B$ with $B \in \mathcal{A}_{q,p}$ and $B_{ji} = A_{ij}^{-1}$ where $1 \leq i \leq p$ and $1 \leq j \leq q$. Let $R \in 2^{\mathcal{A}_{p,q}}$ be, $R^{-1} = \{A^{-1} : A \in R\}$.*
- *Let $R \in 2^{\mathcal{A}_{p,q}}$ and $S \in 2^{\mathcal{A}_{q,r}}$ be, $R \circ S = \prod M$, with $M \in \mathcal{M}(2^{\mathcal{A}_{int}})_{p\times r}$ and $\forall i \in 1,\ldots,p$ and $j \in 1,\ldots,r$, $M_{ij} = \bigcap_{1\leq k\leq q}\{R_{\downarrow ik} \circ S_{\downarrow kj}\}$.*

We can notice that for each atomic relation A and for each relation R, $(A^{-1})^{-1} = A$ and $(R^{-1})^{-1} = R$. The operation of composition defined here is different from the one of [2]. Moreover these fundamental operations satisfy the following properties:
• $X\ R^{-1}\ Y$ iff $Y\ R\ X$,
• If $\exists\ Z$ such that $X\ R\ Z$ and $Z\ S\ Y$ then $X\ R \circ S\ Y$.
Contrary to the composition in IA, the converse of the last implication is not true, for instance let R=$\{(\ b\ )\}$ and S=$\{(\ b \quad eq\ )\}$ be. By considering the two generalized intervals $x = (x_1)$ and $y = (y_1, y_2)$ from the following figure satisfying $R \circ S = \{(\ b \quad b\ )\}$

it is easy to see that there does not exist a 1-interval $z = (z_1)$ satisfying with y the relation $\{(b\ \ eq)\}$ since y_1 and y_2 are equal. We will see in the sequel that it does not matter. For the saturated generalized relations we can prove the following proposition:

Proposition 2 *Let $R, S \in 2^{\mathcal{A}_{p,q}}$ be two saturated relations.*

(a) $R^{-1} = \prod M$, *with* $M \in \mathcal{M}(2^{\mathcal{A}_{int}})_{q\times p}$ *and* $\forall i \in 1,\ldots,p$ *and* $j \in 1,\ldots,q$, $M_{ji} = (R_{\downarrow ij})^{-1}$,
(b) $R \cap S = \prod M$, *with* $M \in \mathcal{M}(2^{\mathcal{A}_{int}})_{p\times q}$ *and* $\forall i \in 1,\ldots,p$ *and* $j \in 1,\ldots,q$, $M_{ij} = R_{\downarrow ij} \cap S_{\downarrow ij}$.

Proof (a) Let $A \in R^{-1}$ be. As $A^{-1} \in R$, $\forall i \in 1,\ldots,p$ and $\forall j \in 1,\ldots,q$, $(A_{ji})^{-1} \in R_{\downarrow ij}$. Hence $A_{ji} \in (R_{\downarrow ij})^{-1}$; it follows that $A \in \prod M$. Let $A \in \prod M$ be. $A_{ji} \in (R_{\downarrow ij})^{-1}$, therefore $(A_{ji})^{-1} \in R_{\downarrow ij}$ and $(A^{-1})_{ij} \in R_{\downarrow ij}$. It follows that $A^{-1} \in \prod R_\downarrow$. Since $R = \prod R_\downarrow$, $A^{-1} \in R$ and $A \in R^{-1}$. (b) Let $A \in R \cap S$ be. $A \in R$ and $A \in S$, consequently $A_{ij} \in R_{\downarrow ij} \cap S_{\downarrow ij}$. This implies that $A \in \prod M$. Let $A \in \prod M$ be. $A_{ij} \in R_{\downarrow ij} \cap S_{\downarrow ij}$, therefore $A_{ij} \in R_{\downarrow ij}$ and $A_{ij} \in S_{\downarrow ij}$. Hence, $A \in \prod R_\downarrow$ and $A \in \prod S_\downarrow$. As R and S are saturated we deduce that $A \in R$ and $A \in S$. □

3 The Convex Relations

In this section and the following one we will define two particular subsets of generalized relations: the set of the convex relations and the set of the preconvex relations. For this purpose we will extend some notions introduced by Ligozat [7, 8] to redefine the convex relations [9] and the ORD-Horn relations [10] (called preconvex relations by Ligozat) of IA. Ligozat arranges the atomic relations of $\mathcal{A}_{int}$ in a partial order which defines a lattice: the interval lattice (see fig. 2). From this order we organize the atomic relations of $\mathcal{A}_{p,q}$ in a partial order $\sqsubseteq$: let $A, B \in \mathcal{A}_{p,q}$ be,

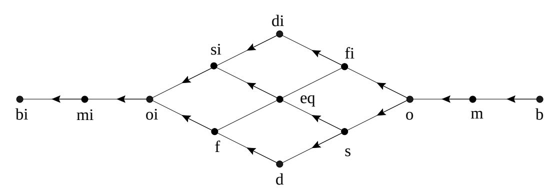


Figure 2. The interval lattice $(\mathcal{A}_{int}, \leq)$.

$$A \sqsubseteq B \text{ iff } \forall i \in 1,\ldots,p \text{ and } \forall j \in 1,\ldots,q,\ \ A_{ij} \leq B_{ij}.$$

$(\mathcal{A}_{p,q}, \sqsubseteq)$ defines a lattice too, called the generalized (p,q)-lattice. Since $(\mathcal{A}_{p,q}, \sqsubseteq)$ is the product order of $(\mathcal{A}_{int}, \leq)$ each interval of the (p,q)-lattice corresponds to a Cartesian product of $p \times q$ intervals of the interval lattice:

Proposition 3 *Let $[A, B]$ be an interval of the (p,q)-lattice. We have: $[A, B] = \prod M$, with $M \in \mathcal{M}(2^{\mathcal{A}_{int}})_{p\times q}$ and $\forall i \in 1,\ldots,p$ and $j \in 1,\ldots,q$, $M_{ij} = [A_{ij}, B_{ij}]$.*

Now we extend the definition of convex closure in the following way:

Definition 2 *Let $R \in 2^{\mathcal{A}_{p,q}}$. The convex closure of R, denoted by $I(R)$, is the relation of $2^{\mathcal{A}_{p,q}}$ corresponding to the smallest interval of the generalized (p,q)-lattice containing R.*

$I(R)$ always exists because the intersection of two intervals of the (p,q)-lattice is an interval too. We have the following properties:

Proposition 4 *Let $R, S \in 2^{\mathcal{A}_{p,q}}$ and $T \in 2^{\mathcal{A}_{q,r}}$ be.*

(a) $R \subseteq I(R)$ *and* $I(I(R)) = I(R)$*,*
(b) *if* $R \subseteq S$ *then* $I(R) \subseteq I(S)$*,*
(c) $I(R) = \prod M$*, with* $M \in \mathcal{M}(2^{\mathcal{A}_{int}})_{p\times q}$ *and* $\forall i \in 1,\dots,p$ *and* $j \in 1,\dots,q$*,* $M_{ij} = I(R_{\downarrow ij})$*,*
(d) $I(R^{-1}) = I(R)^{-1}$*,*
(e) $I(R \circ T) \subseteq I(R) \circ I(T)$*,*
(f) $I(I(R) \cap I(S)) = I(R) \cap I(S)$*,*
(g) $I(I(R) \circ I(T)) = I(R) \circ I(T)$*.*

Proof
$-$ (a) and (b) follows directly from the definition of the convex closure.
$-(c)$: let us denote the interval $I(R_{\downarrow ij})$ of the interval lattice by $[A'_{ij}, B'_{ij}]$ with $A'_{ij}, B'_{ij} \in \mathcal{A}_{int}$. Let us denote by A and B the atomic relations of $\mathcal{A}_{p,q}$ defined by $A_{ij} = A'_{ij}$ and $B_{ij} = B'_{ij}$. From prop. 3 $\prod M = [A, B]$. As $R_{\downarrow ij} \subseteq I(R_{\downarrow ij})$ we deduce that $R \subseteq \prod M$. Consequently, $\prod M$ is an interval of the generalized (p,q)-lattice containing R. Now, we must show that $\prod M$ is the smallest interval containing R. Let a relation $S = [C, D] \in 2^{\mathcal{A}_{p,q}}$ be such that $R \subseteq S$. From prop. 1 (b), $R_{\downarrow ij} \subseteq S_{\downarrow ij}$. It follows that $I(R_{\downarrow ij}) \subseteq I(S_{\downarrow ij})$, but $I(S_{\downarrow ij}) = S_{\downarrow ij}$ because $S_{\downarrow ij}$ is an interval of the interval lattice (prop. 3). Thus, $I(R_{\downarrow ij}) \subseteq S_{\downarrow ij}$, from it we deduce that $\prod M \subseteq \prod S$. As S is a saturated relation (prop. 3) we conclude that $\prod M \subseteq S$.
$-(d)$ follows from (c) and prop. 2.
$-(e)$: from (c) and def. 1 $I(R \circ T) = \prod M$, with $M \in \mathcal{M}(2^{\mathcal{A}_{int}})_{p\times r}$ and $M_{ij} = I(\bigcap_{1\leq k\leq q}\{R_{\downarrow ik} \circ T_{\downarrow kj}\})$. Still from (c) and def. 1, $I(R) \circ I(T) = \prod N$, with $N \in \mathcal{M}(2^{\mathcal{A}_{int}})_{p\times r}$ and $N_{ij} = \bigcap_{1\leq k\leq q}\{I(R_{\downarrow ik}) \circ I(T_{\downarrow kj})\}$. From (a), $R_{\downarrow ik} \circ T_{\downarrow kj} \subseteq I(R_{\downarrow ik} \circ T_{\downarrow kj})$. It follows that $\bigcap_{1\leq k\leq q}\{R_{\downarrow ik} \circ T_{\downarrow kj}\} \subseteq \bigcap_{1\leq k\leq q} I(R_{\downarrow ik} \circ T_{\downarrow kj})$. From (b), $I(\bigcap_{1\leq k\leq q}\{R_{\downarrow ik} \circ T_{\downarrow kj}\}) \subseteq I(\bigcap_{1\leq k\leq q} I(R_{\downarrow ik} \circ T_{\downarrow kj}))$. As the intersection of two intervals of the interval lattice is an interval too, we have $I(\bigcap_{1\leq k\leq q} I(R_{\downarrow ik} \circ T_{\downarrow kj})) = \bigcap_{1\leq k\leq q} I(R_{\downarrow ik} \circ T_{\downarrow kj})$. It follows that $I(\bigcap_{1\leq k\leq q}\{R_{\downarrow ik} \circ T_{\downarrow kj}\}) \subseteq \bigcap_{1\leq k\leq q} I(R_{\downarrow ik} \circ T_{\downarrow kj})$. In IA the property is true, consequently we can conclude that $M_{ij} \subseteq N_{ij}$ therefore $\prod M \subseteq \prod N$.
$-(f)$ results from the fact that the intersection of two intervals of the (p,q)-lattice is also an interval.
$-(g)$: let us show that $I(R) \circ I(T)$ is an interval of the (p,r)-lattice. From (c) and def. 1, $I(R) \circ I(T) = \prod M$, with $M \in \mathcal{M}(2^{\mathcal{A}_{int}})_{p\times r}$ and $M_{ij} = \bigcap_{1\leq k\leq q}\{I(R_{\downarrow ik}) \circ I(T_{\downarrow kj})\}$. Moreover we know that the intersection and the composition of two relations corresponding to two intervals of the interval lattice is also an interval of the interval lattice. From this, we deduce that $M_{ij} = [A'_{ij}, B'_{ij}]$ with $A'_{ij}, B'_{ij} \in \mathcal{A}_{int}$. Let $A, B \in \mathcal{A}_{p,r}$ be defined by $A_{ij} = A'_{ij}$ and $B_{ij} = B'_{ij}$, from prop. 3 $\prod M = [A, B]$. Consequently $I(R) \circ I(T)$ is an interval of the (p,r)-lattice. □

The convex relations of IA [7, 9] correspond to the intervals of the lattice interval. In a natural way and like in [2], we define the convex relations of $2^{\mathcal{A}_{p,q}}$ to be the relations of $2^{\mathcal{A}_{p,q}}$ corresponding to the intervals of the generalized (p,q)-lattice. From prop. 3 we can assert that each convex relation R is a saturated relation and for all $i \in 1,\dots,p$, $j \in 1,\dots,q$, $R_{\downarrow ij}$ is a convex relation of $2^{\mathcal{A}_{int}}$. Obviously $R \in 2^{\mathcal{A}_{p,q}}$ is convex iff $I(R) = R$. From this and prop. 4 we can prove the following theorem:

Theorem 1 *The set of the convex relations of $2^{\mathcal{A}_{p,q}}$ is closed with respect to the fundamental operations $\cap$, $\circ$ and $^{-1}$.*

4 The Weakly-Preconvex Relations

Another important concept of IA is the dimension of a relation. Ligozat represents an interval $x = (x^-, x^+)$ in the real Euclidean plane by a point of coordinates (x^-, x^+). Given a point (x_0^-, x_0^+) representing a reference interval x_0, an atomic relation A of $\mathcal{A}_{int}$ is represented by the region: $\{(y^-, y^+) \in \mathbb{R}^2 : (y^-, y^+)\ A\ (x_0^-, x_0^+)\}$. The resulting regions are: a point (for eq), some semi-lines (for m, mi, f, fi, s, si) and regions of dimension 2 (for b, bi, d, di, o, oi). The dimension of A, denoted by $dim(A)$, is the dimension of the region representing it (see fig. 1). Given a relation $R \in 2^{\mathcal{A}_{int}}$, $dim(R) = max\{dim(A) : A \in R\}$. We define the dimension of a generalized relation in the following way:

Definition 3 *Let $A \in \mathcal{A}_{p,q}$ and $R \in 2^{\mathcal{A}_{p,q}}$ be.* $dim(A) = \sum_{1\leq i\leq p, 1\leq j\leq q} dim(A_{ij})$, $dim(R) = max\{dim(A) : A \in R\}$.

For a saturated generalized relation we have:

Proposition 5 *Let R be a saturated relation of $2^{\mathcal{A}_{p,q}}$.* $dim(R) = \sum_{1\leq i\leq p, 1\leq j\leq q} dim(R_{\downarrow ij})$.

Now, it is time to extend the notion of preconvexity:

Definition 4 *Let $R \in 2^{\mathcal{A}_{p,q}}$ be. R is weakly-preconvex iff* $dim(I(R) \setminus R) < dim(R)$.

Intuitively, a relation is weakly-preconvex iff to compute its convex closure, we only add its atomic relations of dimension strictly lower than its own dimension. Let us notice that the weakly-preconvex relations of $2^{\mathcal{A}_{1,1}}$ are the preconvex relations of IA. For example, let $R = \{\begin{pmatrix} m & m \\ b & b \end{pmatrix}, \begin{pmatrix} o & o \\ b & b \end{pmatrix}\}$ be. R is weakly-preconvex. because $I(R) = \{\begin{pmatrix} m & m \\ b & b \end{pmatrix}, \begin{pmatrix} m & o \\ b & b \end{pmatrix}, \begin{pmatrix} o & m \\ b & b \end{pmatrix}, \begin{pmatrix} o & o \\ b & b \end{pmatrix}\}$, thus $dim(R) = 8$ and $dim(I(R) \setminus R) = 7$.
We extended the concept of preconvexity to the generalized interval algebra in a different way than the one proposed by Balbiani *et al.* in [2]. With their extension the resulting “preconvex” generalized relations – which we will call the saturated-preconvex relations – correspond to the saturated generalized relations whose projections are preconvex. Our notion of weakly-preconvexity subsumes that one:

Proposition 6 *Let $R \in 2^{\mathcal{A}_{p,q}}$ be. if R is saturated-preconvex then R is weakly-preconvex.*

Proof (sketch) Let R be a saturated-preconvex relation and let $A \in I(R)$ be. If $A \notin R$ and $dim(A) \geq dim(R)$ then from prop. 4 (c) and prop. 5 it follows that a projection of R is not preconvex. □

5 The Generalized Networks

Information between several generalized intervals is represented by a special binary CSP: a network of generalized intervals. A network of generalized intervals $\mathcal{N}$ is a structure (V, C) where V is a set of variables $V_1, \dots, V_l$ (with $l = |V|$) ranging over generalized intervals, and where C is a mapping from $V \times V$ onto the set of generalized relations which corresponds to the binary constraints between the generalized intervals. In the sequel, we will denote sometimes by C_{ij} the relation $C(V_i, V_j)$. C is such that:

- $\forall i, j \in 1,\dots,|V|$, $C_{ij} \in 2^{\mathcal{A}_{p,q}}$, with V_i and V_j being respectively a p-interval and a q-interval.

- $\forall i, j \in 1, \ldots, |V|$, $C_{ij} = C_{ji}^{-1}$.
- $\forall i \in 1, \ldots, |V|$, $\forall A \in C_{ii}$, we have $\forall k \in 1, \ldots, p$, $A_{kk} = eq$ (with $C_{ii} \in 2^{\mathcal{A}_{p,p}}$).

A network whose variables represent 1-intervals is an Allen's interval network. With the help of the relation C_{ii} we can constrain the structure of the p-interval represented by V_i. If we want a Ligozat's generalized interval, we just need to take for C_{ii} the following relation of $2^{\mathcal{A}_{p,p}}$:

$$C_{ii} = \{ \begin{pmatrix} eq & m & b & \ldots & b \\ mi & eq & \ddots & \ddots & \vdots \\ bi & \ddots & \ddots & \ddots & b \\ \vdots & \ddots & \ddots & eq & m \\ bi & \ldots & bi & mi & eq \end{pmatrix} \}.$$

To take into account a Ladkin's generalized interval the relation C_{ii} must be the following relation:

$$C_{ii} = \{ \begin{pmatrix} eq & b & \ldots & b \\ bi & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & b \\ bi & \ldots & bi & eq \end{pmatrix} \}.$$

For a n-block, C_{ii} will be the relation composed by all the atomic relations of $\mathcal{A}_{n,n}$ having only the atomic relation eq onto their descending diagonal. So, with the help of a generalized network we can express a constraint network of Ladkin's and Ligozat's generalized intervals, as well as n-block networks.

Definition 5 *Let $\mathcal{N} = (V, C)$ be a network.*

- *$\mathcal{N}$ is said to be saturated (resp. convex, weakly-preconvex) iff all its constraints are saturated (resp. convex, weakly-preconvex).*
- *A consistent instantiation m of $\mathcal{N}$ is a mapping which associates to each variable $V_i \in V$ representing a p-interval, a p-interval noted $m(V_i)$ such that $m(V_i)\ C_{ij}\ m(V_j)$, $\forall i, j \in 1, \ldots, |V|$. The atomic relation satisfied between $m(V_i)$ and $m(V_j)$ will be denoted $m(V_i, V_j)$.*
- *A consistent instantiation m is maximal iff $dim(m(V_i, V_j)) = dim(C_{ij})$ for every $i, j \in 1, \ldots, |V|$.*
- *$\mathcal{N}$ is consistent iff it admits a consistent instantiation.*
- *$\mathcal{N}$ is path-consistent iff for every $i, j, k \in 1, \ldots, |V|$, $C_{ij} \subseteq C_{ik} \circ C_{kj}$ and $C_{ij} \neq \{\}$.*
- *$\mathcal{N}$ is weakly path-consistent iff for every $i, j, k \in 1, \ldots, |V|$, $C_{ij} \subseteq I(C_{ik} \circ C_{kj})$ and $C_{ij} \neq \{\}$.*

Two networks $\mathcal{N} = (V, C)$ and $\mathcal{N}' = (V, C')$ are equivalent iff they have the same consistent instantiations. Like in IA, the problem to know whether a generalized network is consistent is a NP-complete problem in the general case. As we will see in the following section, by using only relations from some subsets of the generalized algebra this problem becomes polynomial. Beforehand let us do some reminders about the well-known path-consistency method and the weak path-consistency method introduced in [4]. Given a network $\mathcal{N} = (V, C)$ the path-consistency method consists of transforming $\mathcal{N}$ into an equivalent network, either being path-consistent or having empty constraints, by iterating the triangulation operation: $C_{ij} \leftarrow C_{ij} \cap (C_{ik} \circ C_{kj})$ until a fixed point is reached. This method can be implemented by an algorithm of complexity $O(|V|^3)$ in time. For the consistency problem this method is sound but not complete: if the empty relation is a constraint of the resulting network then the initial network is inconsistent, else we cannot assert the consistency of the initial network because we are not sure that all the unsatisfiable atomic relations have been removed. The weak path-consistency method is a "weak release" of the path-consistency method. It consists in iterating the weak triangulation operation: $C_{ij} \leftarrow C_{ij} \cap I(C_{ik} \circ C_{kj})$ instead of the usual triangulation one. The former (the weak one) removes less atomic relations than the latter because $R \subseteq I(R)$. It follows that the weak path-consistency method is also sound and not complete. It can be implemented in $O(|V|^3)$ too. After the weak path-consistency method application we obtain an equivalent weakly path-consistent (or empty) network.

6 Tractable Cases

Now we define the projection of a generalized network:

Definition 6 *Let $\mathcal{N} = (V, C)$ be a generalized network. $\mathcal{N}_\downarrow$ is the interval network (V', C') such that:*

- *for each variable $V_i \in V$ representing a p-interval, p variables $V_i^1, \ldots, V_i^p$ belong to V'. V_i^j is the variable which represents the j^{th} subinterval of V_i ;*
- *let V_i^k and $V_j^l \in V'$ be. The constraint $C'(V_i^k, V_j^l)$ is the relation $(C_{ij})_{\downarrow kl}$ of the interval algebra.*

We can prove the following proposition:

Proposition 7 *Let $\mathcal{N} = (V, C)$ be a generalized network. If $\mathcal{N}$ is path-consistent then $\mathcal{N}_\downarrow$ is path-consistent.*

Proof Let $i, j, k \in 1, \ldots, |V|$. Let us now suppose that $C_{ik} \in 2^{\mathcal{A}_{p,q}}$, $C_{kj} \in 2^{\mathcal{A}_{q,r}}$ and let $m \in 1, \ldots, p$ and $n \in 1, \ldots, r$ be. As $C_{ij} \subseteq C_{ik} \circ C_{kj}$, from prop. 1 (b) we can deduce that: $(C_{ij})_{\downarrow mn} \subseteq (C_{ik} \circ C_{kj})_{\downarrow mn}$. From def. 1, it follows that $(C_{ij})_{\downarrow mn} \subseteq \bigcap_{1 \leq l \leq q} \{(C_{ik})_{\downarrow ml} \circ (C_{kj})_{\downarrow ln}\}$. Consequently we have $(C_{ij})_{\downarrow mn} \subseteq (C_{ik})_{\downarrow ml} \circ (C_{kj})_{\downarrow ln}$. Hence $\mathcal{N}_\downarrow$ is path-consistent. □

Concerning the saturated generalized networks we have:

Proposition 8 *Let $\mathcal{N}$ be a saturated generalized network. For each (maximal) consistent instantiation m of $\mathcal{N}$ we can build a (maximal) consistent instantiation of $\mathcal{N}_\downarrow$ and reciprocally.*

Proof Let $\mathcal{N} = (V, C)$ and $\mathcal{N}_\downarrow = (V', C')$ be.
− Let m be a consistent instantiation of $\mathcal{N}$. Let us denote $m(V_i)^k$ the k^{th} interval of $m(V_i)$ associated to the k^{th} subinterval of the p-interval represented by $V_i \in V$ (with $1 \leq k \leq p$). Let m' be the instantiation of $\mathcal{N}_\downarrow$ which associates to each variable $V_i^k \in V'$ the interval $m(V_i)^k$. Let V_i^k and $V_j^l \in V'$ be. Since $m(V_i, V_j) \in C_{ij}$ we deduce that $m'(V_i^k, V_j^l) = (m(V_i, V_j))_{kl} \in (C_{ij})_{\downarrow kl}$, m' is a consistent instantiation of $\mathcal{N}_\downarrow$. Moreover, if $dim(m(V_i, V_j)) = dim(C_{ij})$, as C_{ij} is saturated we have $dim((m(V_i, V_j))_{kl}) = dim((C_{ij})_{\downarrow kl})$. Consequently, if m is maximal then m' is maximal too.
− Let m' be a consistent instantiation of $\mathcal{N}_\downarrow$. Let m be the instantiation of $\mathcal{N}$ defined by: let $V_i \in V$ represent a p-interval, $m(V_i)^k = m'(V_i^k)$ with $k \in 1, \ldots, p$ and $V_i^k \in V'$. Let V_i, $V_j \in V$ represent respectively a p-interval and a q-interval. Let $A \in 2^{\mathcal{A}_{p,q}}$ be defined by $A_{kl} = m'(V_i^k, V_j^l)$. We have $m(V_i, V_j) = A$. Since $m'(V_i^k, V_j^l) \in (C_{ij})_{\downarrow kl}$ and C_{ij} is saturated, $A \in C_{ij}$. Thus m is a consistent instantiation of $\mathcal{N}$. Now let us suppose that $dim(m'(V_i^k, V_j^l)) = dim((C_{ij})_{\downarrow kl})$. As C_{ij} is saturated we deduce that $dim(A) = dim(C_{ij})$. □

Ligozat proved that each convex path-consistent network of IA admits a maximal consistent instantiation. From this we deduce:

Theorem 2 *Let $\mathcal{N}$ be a convex generalized network. If $\mathcal{N}$ is path-consistent then $\mathcal{N}$ admits a maximal consistent instantiation.*

Proof From prop. 3 we deduce that $\mathcal{N}$ is saturated and $\mathcal{N}_{\downarrow}$ is convex. If $\mathcal{N}$ is path-consistent then $\mathcal{N}_{\downarrow}$ is path-consistent (prop. 7). Hence $\mathcal{N}_{\downarrow}$ owns a maximal consistent instantiation. From prop. 8, we conclude that $\mathcal{N}$ admits also a maximal consistent instantiation. □

We extend the convex closure to the generalized networks:

Definition 7 *Let $\mathcal{N}$ be a generalized network. The convex closure of $\mathcal{N}$, denoted by $I(\mathcal{N})$, is the generalized network (V', C') defined by $V' = V$ and $C'_{ij} = I(C_{ij})$.*

We can easily note that the convex closure of a network is always a convex network. Moreover, we have the following property:

Proposition 9 *Let $\mathcal{N}$ be a generalized network. If $\mathcal{N}$ is weakly path-consistent then $I(\mathcal{N})$ is path-consistent.*

Proof If $\mathcal{N}$ is weakly path-consistent then $C_{ij} \subseteq I(C_{ik} \circ C_{kj})$. From prop. 4 (b) and (a) we have $I(C_{ij}) \subseteq I(C_{ik} \circ C_{kj})$. It follows that $I(C_{ij}) \subseteq I(C_{ik}) \circ I(C_{kj})$ (prop. 4 (e)). □

From this result we can prove the following proposition:

Proposition 10 *Each weakly-preconvex and weakly path-consistent generalized network $\mathcal{N}$ admits a maximal consistent instantiation.*

Proof From prop. 9, $I(\mathcal{N})$ is path-consistent, and consequently admits a maximal consistent instantiation m (th. 2). We have $dim(m_{ij}) = dim(I(C_{ij}))$. Since C_{ij} is weakly path-consistent we deduce that $m_{ij} \in C_{ij}$ and $dim(m_{ij}) = dim(C_{ij})$. □

From all this we can prove the following theorem:

Theorem 3 *Let $\mathcal{E}$ be a set of weakly-preconvex generalized relations such that for each relation $R \in 2^{\mathcal{A}_{p,q}}$ belonging to $\mathcal{E}$ and for each convex relation $S \in 2^{\mathcal{A}_{p,q}}$ we have $R \cap S \in \mathcal{E}$. The weak path-consistency method is complete for the consistency problem of the generalized networks whose constraints belong to $\mathcal{E}$.*

Proof Let $\mathcal{N}$ be a generalized network having its constraints in $\mathcal{E}$. By applying the weak path-consistency method to $\mathcal{N}$ we obtain a network $\mathcal{N}'$. If $\mathcal{N}'$ contains the empty relation then $\mathcal{N}$ is inconsistent, else $\mathcal{N}'$ is weakly path-consistent and its constraints belong to $\mathcal{E}$ because $\mathcal{E}$ is stable for the intersection with the convex relations. From prop. 10 we deduce that $\mathcal{N}'$ and $\mathcal{N}$ are consistent. □

In this theorem we can replace the weak path-consistency method by the path-consistency method. Indeed, we can prove that by applying the path-consistency method to a network whose constraints belong to such a set $\mathcal{E}$, we obtain a subnetwork of a weakly path-consistent (or empty) and weakly-preconvex network which is moreover equivalent to the initial network. Using this last theorem we are able to define a tractable set larger than the set of the saturated-preconvex relations: the set of the strongly-preconvex relations.

7 The strongly-preconvex relations

The definition of a strongly-preconvex generalized relation is directly inspired by theorem 3 :

Definition 8 *Let $R \in 2^{\mathcal{A}_{p,q}}$ be. R is strongly-preconvex iff for each convex relation $S \in 2^{\mathcal{A}_{p,q}}$, $R \cap S$ is a weakly-preconvex relation.*

We will denote by $\mathcal{S}$ the set of the strongly-preconvex relations. Now, let us prove that $\mathcal{S}$ satisfies the requirements of theorem 3.

Proposition 11 *Let R be a strongly-preconvex relation of $2^{\mathcal{A}_{p,q}}$. (a) R is a weakly-preconvex relation, (b) $R \cap S \in \mathcal{S}$ for each convex relation S of $2^{\mathcal{A}_{p,q}}$.*

Proof

– The total relation $2^{\mathcal{A}_{p,q}}$ is convex. Hence, $R \cap 2^{\mathcal{A}_{p,q}} = R$ is a weakly-preconvex relation.

– Let S be a convex relation of $2^{\mathcal{A}_{p,q}}$. We must prove that $R \cap S \in \mathcal{S}$. Let T be a convex relation of $2^{\mathcal{A}_{p,q}}$, $(R \cap S) \cap T = R \cap (S \cap T)$. From th. 1 we can deduce that $S \cap T$ is also a convex relation of $2^{\mathcal{A}_{p,q}}$. As R is strongly-preconvex it follows that $R \cap (S \cap T)$ is a weakly-preconvex relation. Hence $R \cap S$ is strongly-preconvex. □

Hence, by applying theorem 3, the consistency problem of strongly-preconvex networks is polynomial. It is easy to see that $\mathcal{S}$ is the largest set to which we can apply this theorem.

8 Conclusion

We defined a very generic framework which subsumes several previous formalisms extending IA. By extending some concepts like dimension and convex closure we characterized a tractable set: the set of the strongly-preconvex relations. Several questions remain open: is the set of the strongly-preconvex generalized relations maximal tractable ? Are there tractable sets (containing the atomic relations) ? Recently, we proved that the set of the weakly-preconvex relations of $2^{\mathcal{A}_{p,q}}$ (with $p, q \leq 2$) is not tractable. For that purpose we exhibited a polynomial reduction from the 3-coloring graph problem to the consistency problem of the weakly-preconvex generalized networks. It is a beginning of an answer to the former question. The path-consistency method and the weak path-consistency method are complete for the set of the strongly-preconvex generalized networks. Currently, we study the advantages and drawbacks of these methods, one w.r.t. the other.

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