A Foundation for Region-Based Qualitative Geometry[1]

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Abstract. We present a highly expressive logical language for describing qualitative configurations of spatial regions, based on Tarski's *Geometry of Solids*, in which the *parthood* relation and the concept of *sphere* are taken as primitive. We give a categorical axiom system, whose models can be interpreted classically in terms of Cartesian spaces over $\mathbb{R}$. We show that within this system the concept of sphere and the *congruence* relation are interdefinable. We investigate the 2nd-order character of the theory and prove incompleteness of some weaker 1st-order variants.

1 INTRODUCTION

Many researchers in the field of Qualitative Spatial Reasoning (QSR) have argued that it is useful to have representations in which *spatial regions* are the basic entities [7, 5]. This ontology contrasts with the approach of classical geometry, where lines, surfaces and regions are typically thought of as sets of points. To meet this need several region-based theories have been proposed [16, 1]. However, these theories have been limited to describing topological properties, so the expressive power is much more restricted than point-based geometry.

By adding a *sphere* primitive to Leśneiwski's *Mereology*, Tarski [19] showed how to give a categorical axiomatisation of the geometry of regions whose models are isomorphic to the structure of regular open sets of points in Euclidean point-based geometry. He called this theory the *Geometry of Solids*.[3] Unfortunately Tarski's theory is not fully formalised: its postulates are stated in English and in several cases are meta-level conditions rather than axioms. The current paper employs Tarski's ideas to provide a fully formal system.

One paper which does propose a region based theory with the full expressive power of geometry is [18], which attempts to reconstruct Tarski's theory within a 1st-order language with the primitives of *congruence* and *strong connection.* Our approach is similar to this but our theory has some significant advantages. Firstly, (as in [6]) we assume only *parthood,* $\mathsf{P}(x, y)$, and a morphological primitive (which may be either the sphere predicate $\mathsf{S}(x)$ or the *congruence* relation $\mathsf{CG}(x, y)$), whereas [18] employ an additional topological primitive ('simple region'). Topological concepts are nevertheless definable in our system. Secondly, we prove categoricity by a much more direct encoding of Tarski's geometry axioms into our language.

There has recently been some criticism of the region-based approach to spatial reasoning. For instance [15] argue that there is no advantage in reasoning with regions as opposed to sets of points because, once a certain minimal level of expressive capability is passed, the region-based theories are just as complex as comparable point-based theories. While we accept this, we believe that for carrying out reasoning relevant to specific problems involving configurations of spatial regions there are significant advantages in using a representation which is close to the obvious region-based description of the problem, rather than re-describing the situation in terms of points. In fact the high complexity of reasoning in any expressive spatial language makes it all the more important that one should be able to specify problems as simply as possible.

The primary purpose of our formalism is to provide a secure ontological foundation (as advocated e.g. in [13]) for theories of spatial information; but we also believe that it can serve as a framework within which more computationally oriented representations (e.g. that of [17]) can be embedded. Since our theory has a categorical interpretation in terms of Cartesian fields over $\mathbb{R}$ it is readily compatible with more traditional representations that employ this classical model of space.

2 MEREOLOGY

We begin by presenting a formal theory of the parthood relation, $\mathsf{P}(x, y)$. As a basis for the axiomatisation we take the classical Mereology of Leśneiwski [14] (see also [19, 21]):

D1) $\mathsf{PP}(x, y) \equiv_{def} (\mathsf{P}(x, y) \wedge \neg(x = y))$
D2) $\mathsf{DR}(x, y) \equiv_{def} \neg\exists z[\mathsf{P}(z, x) \wedge \mathsf{P}(z, y)]$
D3) $\mathsf{SUM}(\alpha, x) \equiv_{def} \forall y[y \in \alpha \rightarrow \mathsf{P}(y, x)] \wedge \neg\exists z[\mathsf{P}(z, x) \wedge \forall y[y \in \alpha \rightarrow \mathsf{DR}(y, z)]]$

In **D3**, α is a 2nd-order variable, which can denote any subset of the domain of regions. $x \in \alpha$ is of course true just in case the denotation of x is a member of the set denoted by α; but our object language does not include any other set-theoretic apparatus.[4]

In addition to the usual principles of classical logic and the theory of sets, the system is required to satisfy the following specifically mereological postulates:

A1) $\forall x\forall y\forall z[\mathsf{P}(x, y) \wedge \mathsf{P}(y, z) \rightarrow \mathsf{P}(x, z)]$
A2) $\forall\alpha[\exists x[x \in \alpha] \rightarrow \exists! x[\mathsf{SUM}(\alpha, x)]]$

These ensure firstly that the part relation is transitive and secondly (and slightly controversially) that for any non-empty set of individuals there is a unique individual which is the sum of that set.

For convenience we also define:

D4) $\mathsf{O}(x, y) \equiv_{def} \neg\mathsf{DR}(x, y)$
D5) $\mathsf{PO}(x, y) \equiv_{def} \mathsf{O}(x, y) \wedge \neg\mathsf{P}(x, y) \wedge \neg\mathsf{P}(y, x)$

3 REGION-BASED GEOMETRY

We now develop a theory which we call *Region-Based Geometry*, inspired by Tarski's Geometry of Solids [19]. Whereas Tarski's pre-

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[3] This is really a theory of 'regions' or 'volumes', since the entities of the theory may inter-penetrate and the property of solidity is not considered.

[4] In fact the form $x \in \alpha$ could be written as $\alpha(x)$, in the style of a 2nd-order language without set theory.

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sentation is not formalised (and in some respects somewhat unclear) we shall give a fully formal axiomatisation.

Following Tarski we build on Leśniewski's *mereology* by introducing a new primitive *sphere* predicate, which we write $\mathsf{S}(x)$. In terms of P and S a series of geometrical relationships and concepts are defined and a set of postulates is given. Here and in the rest of the paper we shall often want to quantify over just the spherical regions in the domain. For convenience we introduce the notations

- $\forall^\circ x[\phi] \;\equiv_{def}\; \forall x[\mathsf{S}(x) \to \phi]$
- $\exists^\circ x[\phi] \;\equiv_{def}\; \exists x[\mathsf{S}(x) \wedge \phi]$

As in [19] we define the relations of *external tangency* (ET), *internal tangency* (IT), *external diametricity* (ED), *internal diametricity* (ID) and *concentricity* ($x \circledcirc y$). See Fig. 1 for 2D illustrations.

D6) $\mathsf{ET}(a,b) \;\equiv_{def}\; (\mathsf{S}(a) \wedge \mathsf{S}(b) \wedge \mathsf{DR}(a,b) \wedge \forall^\circ xy[(\mathsf{P}(a,x) \wedge \mathsf{P}(a,y) \wedge \mathsf{DR}(b,x) \wedge \mathsf{DR}(b,y)) \to (\mathsf{P}(x,y) \vee \mathsf{P}(y,x))])$

D7) $\mathsf{IT}(a,b) \;\equiv_{def}\; (\mathsf{S}(a) \wedge \mathsf{S}(b) \wedge \mathsf{PP}(a,b) \wedge \forall^\circ xy[(\mathsf{P}(a,x) \wedge \mathsf{P}(a,y) \wedge \mathsf{P}(x,b) \wedge \mathsf{P}(y,b)) \to (\mathsf{P}(x,y) \vee \mathsf{P}(y,x))])$

D8) $\mathsf{ED}(a,b,c) \;\equiv_{def}\; (\mathsf{S}(a) \wedge \mathsf{S}(b) \wedge \mathsf{S}(c) \wedge \mathsf{ET}(a,c) \wedge \mathsf{ET}(b,c) \wedge \forall^\circ xy[(\mathsf{DR}(x,c) \wedge \mathsf{DR}(y,c) \wedge \mathsf{P}(a,x) \wedge \mathsf{P}(b,y)) \to \mathsf{DR}(x,y)])$

D9) $\mathsf{ID}(a,b,c) \;\equiv_{def}\; (\mathsf{S}(a) \wedge \mathsf{S}(b) \wedge \mathsf{S}(c) \wedge \mathsf{IT}(a,c) \wedge \mathsf{IT}(b,c) \wedge \forall^\circ xy[(\mathsf{DR}(x,c) \wedge \mathsf{DR}(y,c) \wedge \mathsf{ET}(a,x) \wedge \mathsf{ET}(b,y)) \to \mathsf{DR}(x,y)])$

D10) $a \circledcirc b \;\equiv_{def}\; (\mathsf{S}(a) \wedge \mathsf{S}(b) \wedge ((a = b) \vee (\mathsf{PP}(a,b) \wedge \forall^\circ xy[(\mathsf{ED}(x,y,a) \wedge \mathsf{IT}(x,b) \wedge \mathsf{IT}(y,b)) \to \mathsf{ID}(x,y,b)]) \vee (\mathsf{PP}(b,a) \wedge \forall^\circ xy[(\mathsf{ED}(x,y,b) \wedge \mathsf{IT}(x,a) \wedge \mathsf{IT}(y,a)) \to \mathsf{ID}(x,y,a)])))$

We now define some fundamental relations involving spheres:

D11) $\mathsf{B}(x,y,z) \;\equiv_{def}\; x = y \vee y = z \vee \exists vw[\mathsf{ED}(x,y,v) \wedge \mathsf{ED}(v,w,y) \wedge \mathsf{ED}(y,z,w)]$

D12) $\mathsf{COB}(s,r) \;\equiv_{def}\; \mathsf{S}(s) \wedge \forall s'[s' \circledcirc s \to (\mathsf{O}(s',r) \wedge \neg \mathsf{P}(s',r))]$

D13) $\mathsf{EQD}(x,y,z) \;\equiv_{def}\; \exists^\circ z'[z' \circledcirc z \wedge \mathsf{COB}(y,z') \wedge \mathsf{CB}(x,z')]$

D14) $\mathsf{Mid}(x,y,z) \;\equiv_{def}\; \mathsf{B}(x,y,z) \wedge \exists^\circ y'[y' \circledcirc y \wedge \mathsf{COB}(x,y') \wedge \mathsf{COB}(z,y')]$

D15) $\mathsf{EQD}(w,x,y,z) \;\equiv_{def}\; \exists^\circ uv[\mathsf{Mid}(w,u,y) \wedge \mathsf{Mid}(x,u,v) \wedge \mathsf{EQD}(v,z,y)]$

D16) $\mathsf{Nearer}(w,x,y,z) \;\equiv_{def}\; \exists^\circ x'[\mathsf{B}(w,x,x') \wedge \neg(x \circledcirc x') \wedge \mathsf{EQD}(w,x',y,z)]$

$\mathsf{B}(x,y,z)$ holds when the centre of y is between the centres of x and z (or coincides with one of these). $\mathsf{COB}(s,r)$ means that sphere s is Centred On the Boundary of r. $\mathsf{EQD}(x,y,z)$ says that the centres of x and y are equidistant from the centre of z. $\mathsf{Mid}(x,y,z)$ says that the centre of y lies mid-way between the centres of x and z; and $\mathsf{EQD}(w,x,y,z)$ holds when the distance between the centres of w and x is the same as the distance between the centres of y and z. $\mathsf{Nearer}(w,x,y,z)$ means that the centres of w and x are closer than the centres of y and z.

Since the concepts B and EQD are definable, we can write the axioms of n-dimensional *Elementary Geometry* [20, 2] within our language (the value of n is fixed by appropriate choice of upper and lower dimension axioms). [19] takes this approach to prove that his geometry of solids is categorical and is modelled by n-dimensional Euclidan space in which spheres are interpreted as open balls and 'solids' are regular open sets. We take a similar approach; however, whereas Tarski introduced *points* as sets of spheres, our relations concern spheres but they hold just in case the centre points of the spheres satisfy the corresponding point relations. Thus the quantifiers of the point-based geometry axioms can be replaced by quantifiers over spheres and the equality relation replaced by the $\circledcirc$ relation. Hence, in addition to **A1** and **A2** of Mereology, our theory contains:

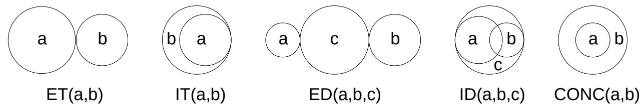

Figure 1. Relations among spheres defined by Tarski [19].

A3) A complete axiom set for n-dimensional geometry (e.g. [20]) encoded in terms of the B, EQD and $\circledcirc$ relations.

A4) $\forall^\circ xyz[(x \circledcirc y \wedge y \circledcirc z) \to x \circledcirc z]$

A5) $\forall^\circ xx'yzw[(\mathsf{EQD}(x,y,z,w) \wedge x' \circledcirc x) \to \mathsf{EQD}(x',y,z,w)]$

Axioms **A4** and **A5** ensure that $\circledcirc$ behaves like equality relative to the geometrical axioms.[5]

To get a categorical axiomatisation we have to ensure that the class of regular open sets of centre points of spheres coincides with the class of regions and that the P relation corresponds to the inclusion relation among the centre points. Rather than stating these as a meta-level constraints (as Tarski does) we enforce them directly by axioms. First, we define relations that hold when the centre point of a sphere is within the interior of a region:

D17) $\mathsf{InI}(s,r) \;\equiv_{def}\; \exists s'[s' \circledcirc s \wedge \mathsf{P}(s',r)]$

We can now specify the axioms:

A6) $\forall^\circ xy[\neg(x \circledcirc y) \to \exists^\circ s[s \circledcirc x \wedge \forall^\circ z[\mathsf{InI}(z,s) \leftrightarrow \mathsf{Nearer}(x,z,x,y)]]$

A7) $\forall^\circ x \exists^\circ y[\neg(x \circledcirc y) \wedge \forall^\circ z[\mathsf{InI}(z,x) \leftrightarrow \mathsf{Nearer}(x,z,x,y)]]$

A8) $\forall xy[\mathsf{P}(x,y) \leftrightarrow \forall^\circ s[\mathsf{InI}(s,x) \to \mathsf{InI}(s,y)]]$

A9) $\forall r \exists^\circ s[\mathsf{P}(s,r)]$

A6 ensures that for every pair of distinct points x and y there is a sphere centred at one and bounded by the other. **A7** says that all spheres can be constructed in this way. **A8** means that $\mathsf{P}(x,y)$ holds just in case every interior point of x is an interior point of y (this actually makes **A1** redundant). **A9** states that every region has a spherical part (from this it can be proofed that every region is equal to the sum of its spherical parts). The theory specified by the axioms **A1–9** we call $\mathbf{RBG}^n$ (Region-Based Geometry).

Let an *n-dimensional classical interpretation* for $\mathbf{RBG}^n$ be a function $\Im_n$ which assigns a *non-empty regular open subset* of $\mathbb{R}^n$ to each 1st-order variable of $\mathbf{RBG}^n$ and a set of non-empty regular open subsets of $\mathbb{R}^n$ to each of its 2nd-order variables. Under a classical interpretation $\mathsf{P}(x,y)$ holds just in case $\Im_n(x) \subseteq \Im_n(y)$; $\mathsf{S}(x)$ holds just in case $\Im_n(x)$ is an open n-ball.

Theorem 1 *Axioms $\mathbf{RBG}^n$ provide a categorical axiom system for n-dimensional region-based geometry, such that every model is isomorphic to a classical interpretation $\Im^n$.*

Proof: [20] proves that all models of the axioms specified by **A3-A5** have the structure of n-dimensional Cartesian spaces over $\mathbb{R}$. This guarantees that there are spheres centred at all and only the points of $\mathbb{R}^n$. **A6** and **A7** ensure that there is a sphere corresponding to every open n-ball in the space, such that $\mathsf{InI}(s,s')$ holds just in case the centre point of s is in the ball corresponding to s'. **A8** fixes the interpretation of $\mathsf{P}(x,y)$ to coincide with the condition $\{s|\mathsf{InI}(s,x)\} \subseteq \{s|\mathsf{InI}(s,y)\}$. The interpretation of S is now completely fixed relative to $\mathbb{R}^n$ and InI and P also have there intended interpretations over the domain of spheres. We now fix the interpretation of the regions.

[5] Reflexivity and symmetry are implicit in the definition of $\circledcirc$.

From **A9** and **D3** it is easy to show that every region is the sum of its spherical parts (**T1**: $\forall x[\mathsf{SUM}(\{y \mid \mathsf{P}(y,x) \wedge \mathsf{S}(y)\}, x)]$). This, with **A2**, ensures that the set of regions coincides with the SUMs of arbitrary sets of spheres. Let the set of 'interior-points' of a region r be the set of centre points of all spheres s such that $\mathsf{InI}(s,r)$. Clearly, determining this set for all regions completely determines the P relation. We now show that for any set of spheres α such that $\mathsf{SUM}(\alpha, s)$, the set S of interior points of s coincides with the smallest regular open set R containing all interior points of all spheres in α.

First note that from **A1**, **A2**, **D10** and **D17** one can prove **T2**: $\forall^{\circ} s \forall r[\mathsf{InI}(s,r) \leftrightarrow \exists^{\circ} s'[\mathsf{P}(s',r) \wedge \mathsf{InI}(s,s')]]$; which means that the interior points of a region are just those interior to its spherical parts. Consequently the interior points of any region form an open set. Since S must be open and R is regular open, then if S were larger than R it would contain some sphere disjoint from R and hence disjoint from all spheres in α and hence disjoint from s. Thus $S \subseteq R$. Since R is open we can exactly cover all its points by the interior points of a set of spheres ρ. By **A2** there must be a region r such that $\mathsf{SUM}(\rho, r)$. The regularity of R then means that interior points of r are exactly those in R. Now suppose $S \subsetneq R$, then using **A8** and the mereological axioms one can show that $\exists x[\mathsf{P}(x,r) \wedge \mathsf{DJ}(x,s)]$ and thence (using **9**) $\exists^{\circ} x[\mathsf{P}(x,r) \wedge \neg\mathsf{DJ}(x,s)]$. But if r includes a sphere which is disjoint from s then, contrary to our supposition R cannot be the *smallest* regular open set including all spheres in α. Therefore we must have $S = R$. ■

Because of the 2nd-order nature of the theory we cannot get a truly complete axiom system. However, the categoricity result means the theory would be complete if we had an oracle for 2nd-order logic, so the meaning of all non-logical vocabulary is completely fixed.

Theorem 2 **RBG**^{n} *is undecidable for* $n \geq 2$.

Proof: For $n = 2$ this follows from the results of [12] and also of [9]. Undecidability for higher dimensions can be demonstrated by defining a 'slice' of n-space that is infinitely extended in two dimensions but of a fixed finite thickness in the others. Two dimensional regions can be simulated by parts of the slice such that their boundaries within the slice are orthogonal to the faces of the slice.[6] ■

4 PRIMITIVES AND DEFINABILITY

It is worth noting that all the concepts used in our axiomatisation were defined just from the primitives P and S. Nevertheless, **RBG** is an extremely expressive formalism. All of elementary point geometry can be described by means of the B and EQD predicates and the $\forall^{\circ}$ and $\exists^{\circ}$ quantifiers. Additionally we can define a further very rich class of predicates which apply to regions in general. We can define a *connection* relation (which behaves similarly to the C primitive of [16]) by noting that for every point of contact (or overlap) between two regions, any sphere centred on that point overlaps both regions. We can also define the *strong connection* relation of [18] and the *can connect* primitive of [8]:

D18) $\mathsf{C}(x,y) \equiv_{def} \exists^{\circ} z \forall z'[z' \circledcirc z \rightarrow (\mathsf{O}(z',x) \wedge \mathsf{O}(z',y))]$
D19) $\mathsf{SC}(x,y) \equiv_{def} \exists^{\circ} z[\mathsf{O}(z,x) \wedge \mathsf{O}(z,y) \wedge \mathsf{P}(z, x+y)]$
D20) $\mathsf{CC}(x,y,z) \equiv_{def} \exists x'[\mathsf{CG}(x',x) \wedge \mathsf{C}(x',y) \wedge \mathsf{C}(x',z)]$

We can also define the powerful *congruence* relation employed by [18] and [6]. Two regions are congruent if one can be transformed into the other by means of rotation, translation and (optionally) mirror reflection operations.[7] Congruence of spheres can be defined by:

D21) $\mathsf{SCG}(x,y) \equiv_{def} \mathsf{S}(x) \wedge \mathsf{S}(y) \wedge \exists s_1 s_2[s_1 \circledcirc s_2 \wedge \mathsf{ET}(x,s_1) \wedge \mathsf{ET}(y,s_1) \wedge \mathsf{IT}(x,s_2) \wedge \mathsf{IT}(y,s_2)]$

and of two pairs of spheres by:

D22) $\mathsf{SCG}(x,y,x',y') \equiv_{def} \mathsf{SCG}(x,x') \wedge \mathsf{SCG}(y,y') \wedge ((\mathsf{ET}(x,y) \wedge \mathsf{ET}(x',y')) \vee \exists zz'[\mathsf{SCG}(z,z') \wedge \mathsf{ED}(x,y,z) \wedge \mathsf{ED}(x',y',z')] \vee \exists zz'[\mathsf{SCG}(z,z') \wedge \mathsf{IT}(z,x) \wedge \mathsf{IT}(z,y) \wedge \mathsf{IT}(z',x') \wedge \mathsf{IT}(z',y')])$

We can then define a general congruence relation, $\mathsf{CG}(x,y)$, between arbitrary regions by employing the construction given in [18] using so-called '*scalene* sums of spheres'.

FROM MEREOLOGY+CONGRUENCE TO SPHERES

A significant result of our investigations is that within **RBG** the sphere predicate S is definable in terms of P and CG. Our approach is first to define the relation $\mathsf{CGOB}(x,y,z)$, which means that a region ConGruent to x Overlaps Both of the regions y and z:

D23) $\mathsf{CGOB}(x,y,z) \equiv_{def} \exists x'[\mathsf{CG}(x,x') \wedge \mathsf{O}(x',y) \wedge \mathsf{O}(x',z)]$

Under the intended interpretation where variables range over regular open subsets of $\mathbb{R}^n$ and CG is the ordinary geometrical congruence relation, $\mathsf{CGOB}(x,y,z)$ holds just in case there are points $p_1, p_2 \in x$, $p_y \in y$ and p_z in z, such that $d(p_1,p_2) = d(p_y,p_z)$, where $d(p_x,p_y)$ is the distance between p_x and p_y.

Let us now define an ordering on regions based upon the pairs to which they stand in the CGOB relation, and a predicate which identifies those regions that are maximal with respect to this order:

D24) $x \preceq y \equiv_{def} \forall z_1 z_2[\mathsf{CGOB}(x,z_1,z_2) \rightarrow \mathsf{CGOB}(y,z_1,z_2)]$
D25) $x \prec y \equiv_{def} x \preceq y \wedge \neg(y \preceq x)$
D26) $\mathsf{MaxCGOB}(x) \equiv_{def} \forall y[\mathsf{PP}(x,y) \rightarrow x \prec y]$

It is clear that $\mathsf{MaxCGOB}(x)$ must hold if x is a sphere. Unfortunately there are also some more complex regions for which it holds. E.g., the predicate is satisfied by regions consisting of three congruent spheres arranged so that their centre points form an equilateral triangle, provided that the distance between each pair of spheres is greater than the diameter of the spheres. It is possible to construct still more complex examples, where a MaxCGOB region consists of multiple non-congruent components. Nevertheless, we can prove:

Lemma 1 *In a classical interpretation every* connected *region* r*, such that* $\mathsf{MaxCGOB}(r)$ *holds, is a sphere.*

Proof: For any connected region r, the set of region pairs $\langle x,y \rangle$, such that $\mathsf{CGOB}(r,x,y)$ is completely determined by the separation, δ, of the two most distant points in the closure of r. The largest connected region having a given value of δ must be a sphere. ■

We now define a special class of one-piece regions which also includes spheres as a special case. First we define a relation which holds between a region and the sum of all regions which are both congruent to and overlap that region:

D27) $\mathsf{CGOSUM}(x,y) \equiv_{def} \mathsf{SUM}_z(\mathsf{CG}(z,x) \wedge \mathsf{O}(z,x) : y)$

Since sums always exist and are unique, for every region x there will be a unique region y such that $\mathsf{CGOSUM}(x,y)$. We now prove that:

[6] **RBG**^{1} may also be undecidable; so far we have no result on this.

[7] In [3] we used the term 'congruent' for cases where mirror reflections are excluded and 'isometry' for the more general relation allowing reflections. In the current paper, to accord with published nomenclature (e.g. [18] we use 'congruence' in the more general sense.

Lemma 2 $\exists x[\mathsf{CGOSUM}(x,r)]$ *can be true in a classical interpretation only if* r *is self-connected.*

Proof: We consider three exhaustive cases covering possible instances of the existentially quantified generating region x:

1. x is self connected. Clearly r must be connected.
2. x contains at least one unbounded component. Then r must be the universe, which is self connected.
3. x consists of a number bounded components.

We need to prove the theorem for case 3. Connectendess of this case follows from two lemmas: a) every pair of components of x are connected *via* a continuous path in r; and b) every point of r is connected to some component of x *via* a continuous path in r.

We note that for any point p of the space, $p \in r$ just in case: there are three points $p_1, p_2, p_3 \in x$, such that $d(p, p_1) = d(p_2, p_3)$ (since this means we can translate and rotate the space so that points p_2 and p_3 are mapped to points p and p_1). Moreover, for any three points $p_1, p_2, p_3 \in x$, all points on the sphere centred on p_1 of radius $d(p_2, p_3)$ are in r. We denote the set of points of this sphere by $\sigma(p_1, \langle p_2, p_3\rangle)$.

Now to show lemma a) take any two components c_1, c_2 of x and two points $p_1 \in c_1$ and $p_2 \in c_2$. The two congruent spheres $\sigma(p_1, \langle p_1, p_2\rangle)$ and $\sigma(p_2, \langle p_1, p_2\rangle)$ are both within r; also, each passes through the centre of the other and they must therefore intersect. Clearly there is a continuous path between p_1 and p_2 running along the surface of first one and then the other of these spheres.

Now for lemma b). Take an arbitrary point $p \in r$. This must be in $\sigma_1 = \sigma(p_1, \langle p_2, p_3\rangle)$ for some three points $p_1, p_2, p_3 \in x$; and these points must each be within some component of x. Suppose p_2 lies outside σ_1, then the sphere $\sigma_2 = \sigma(p_2, \langle p_2, p_1\rangle)$ intersects both σ_1 and the point p_1. Thus, there is a path from p to p_1 by traversing σ_1 and then σ_2. Similarly, if p_3 lies outside σ_1 one can find a path by traversing σ_1 then $\sigma(p_3, \langle p_3, p_1\rangle)$. Finally, in the case where both p_2 and p_3 lie inside σ_1, the sphere $\sigma_3 = \sigma(p_3, \langle p_2, p_3\rangle)$ is congruent to σ_1 and centred within σ_1 and so must intersect σ_1; thus we can find a path from p along σ_1 and then along σ_3 to reach p_2. ■

We are now ready to define spheres. It is easy to see that for any sphere s there is a region s' such that $\mathsf{CGOSUM}(s', s)$. Specifically this holds where s' is a sphere concentric with s and with one third of its radius. Thus, whereas $\mathsf{MaxCGOB}(x)$ holds for all spheres and for a class of exotic multi-piece regions, $\exists y[\mathsf{CGOSUM}(y,x)]$ holds for all spheres and a class of exotic self-connected regions. Consequently, we define a *sphere* predicate by:

D28) $\mathsf{S}(x) \equiv_{def} (\mathsf{MaxCGOB}(x) \wedge \exists y[\mathsf{CGOSUM}(y,x)])$

Thus we have proved:

Theorem 3 *The* sphere *predicate is definable in terms of parthood and congruence. More specifically, there is a predicate* S *definable from the relations* P *and* CG*, such that under any classical interpretation* $\mathsf{S}(x)$ *is true iff region* x *is a sphere (in the usual sense).*

5 ALTERNATIVE AXIOMS

Given the interdefinability of many of the key concepts of our axiomatisation it is clear that very different and perhaps more elegant equivalent axiom sets could be given. In particular one might want to replace the complex axioms of Elementary Geometry with more basic properties of qualitative concepts. For instance the lower and upper dimension axioms for a two-dimensional space can be replaced by the simple conditions:

- $\exists^\circ xyz[\mathsf{ET}(x,y) \wedge \mathsf{ET}(y,z) \wedge \mathsf{ET}(x,z)]$
- $\neg\exists^\circ xyzw[\mathsf{ET}(x,y) \wedge \mathsf{ET}(y,z) \wedge \mathsf{ET}(x,z) \wedge \mathsf{ET}(w,x) \wedge \mathsf{ET}(w,y) \wedge \mathsf{ET}(w,z)]$

Drawing on [18] we would like to recast our axioms so that properties of the CG relation play a major rôle. Some strong properties of CG and its relation to P are the following:

CG1) $\mathsf{CG}(x,y)$ is an equivalence relation.
CG2) $\forall xy[\mathsf{PP}(x,y) \rightarrow \neg\mathsf{CG}(x,y)]$
CG3) $\forall xy[\exists x'[\mathsf{CG}(x',x) \wedge \mathsf{P}(x',y)] \leftrightarrow \exists y'[\mathsf{CG}(y',y) \wedge \mathsf{P}(x,y')]]$
CG4) $\forall x \forall y[(x = \mathcal{U}) \vee (y = \mathcal{U}) \vee \exists z[\mathsf{CG}(x,z) \wedge \mathsf{PO}(y,z)]]$

Using the HOL theorem prover [10] we have already verified that several of the simpler geometry axioms giving basic properties of B and EQD can be derived from these congruence axioms.

6 WEAKER 1ST-ORDER VARIANTS

From the point of view of specifying an ontology of spatial entities the theories **RBG**^{n} are appealing because of their categorical interpretation in terms of point-based geomentry. However, as a basis for automated theorem proving they have serious drawbacks. A possible restriction is to limit the construction of sums to sets describable by a 1st-order predicate of the language. Thus we could define

D29) $\mathsf{SUM}_x(\phi(x) : y) \equiv_{def} \forall z[\phi(z) \rightarrow \mathsf{P}(z,y)] \wedge \neg\exists z[\mathsf{P}(z,y) \wedge \forall w[\phi(w) \rightarrow \mathsf{DR}(w,z)]]$

to mean that y is the sum of all regions satisfying the predicate $\phi(x)$. The existence and uniqueness of sums then corresponds to the schematic axiom

A2') $\exists x[\phi(x)] \rightarrow \exists! y[\mathsf{SUM}_x(\phi(x) : y)]$

This schema is not equivalent to any single 1st-order axiom.

For some practical purposes one may want to further restrict the existence of sums axiom so that only sums of finite sets of regions are guaranteed to exist. This can be achieved by replacing **A2** by [8]

A2") $\forall xy \exists! z[z = x + y]$

We call the weaker theories obtained by replacing **A2** by **A2'** or **A2" RBG'**^{n} and **RBG"**^{n}.

Theorem 4 *The theories* **RBG'**^{n} *and* **RBG"**^{n}*, where* $n \geq 2$*, are undecidable.*

Proof (sketch): Using the ideas of [12], the arithmetical concepts of equality, addition and multiplication can be encoded within either **RBG'**^{n} or **RBG"**^{n} as relations among regions, whose number of components is associated with a natural number. One could then extend the theories to include an essentially undecidable set of 1st-order aritmetical axioms. Being subtheories of an essentially undecidable theory **RBG'**^{n} and **RBG"**^{n} must themselves be undecidable. ■

Theorem 5 *The theories* **RBG'**^{n} *and* **RBG"**^{n} *are incomplete.*

Proof: Every complete 1st-order theory axiomatisable by a finite number of finitary axioms and axiom schemata is decidable. Since these theories are undecidable they must be incomplete.[9] ■

It should be noted that even the theories **RBG"**^{n} are not ordinary 1st-order theories because the axioms of Elementary Geometry include a schematic continuity axiom standing for an infinite set of

[8] [18] claim that restriction to binary sums is necessary to avoid contradiction; but this only applies to the definition of sums in terms of connection used by [4].

[9] The same argument shows that the theory of [18] is incomplete.

formulae. It is possible to replace the schema with a single axiom such that the system remains categorical despite being incomplete [20]. However, this weakening is unlikely to be computationally advantageous since the full Elementary Geometry is decidable, whereas the weaker form is thought to be undecidable.

7 THE 2ND-ORDER CHARACTER OF THE THEORY

In general a 2nd-order variable ranges not only over sets of entities falling under a single object language predicate but over entities falling under any of some (possibly infinite) set of such predicates.

In the case of **RBG**, suppose we have proof of $\exists[\phi_0(x)]$ and a sequence of predicates: $\phi_1(x), \phi_2(x), \ldots$; then from **A2** we can derive

$$\exists! y[\forall ix[\phi_i(x) \rightarrow \mathsf{P}(x,y)] \wedge \forall x[\mathsf{P}(x,y) \rightarrow \exists iz[\phi_i(z) \wedge \mathsf{O}(x,z)]]]$$

Here we represent a potentially infinite sequence of predicates by quantification over an indexing variable i. We can then specify (1st-order) constraints on the sequence of predicates by formulae such as $\forall ix[(\phi_i(x) \wedge \psi_1(x)) \rightarrow \exists iy[\phi_i(y) \wedge \psi_2]]$, where ψ_1 and ψ_2 are morpho-mereological predicates that are relevant to some proof.

Restriction to countable sequences of predicates does not limit the generality of the inference rule in so far as only a countable set of predicates could be relevant to the proof of any 1st-order theorem. But there are, of course, an uncountable number of (countable) sequences of predicates. Moreover, because **RBG** is known to be undecidable, the theory must have 1st-order theorems whose proof depends on the construction of predicate sequences satisfying constraints that are not expressible by finite 1st-order formulae.

On the other hand there are interesting theorems that can be proved by means of predicate sequences satisfying simple 1st-order properties. For instance, one can construct a sequence of predicates describing a specific series of disconnected spheres, each twice the diameter of its predecessor; and by constructing the sum of these spheres one can prove that there exists some region r satisfying the conditions: **i**) all components of r are spheres; **ii**) r has a unique smallest component; **iii**) for every component c' of r there is another component c' such that c' has twice the radius of c. These conditions mean that r must have a denumerable infinity of components and, because we can define a relation holding between regions that have an equal number of components (this can be done in the style of [12]), this means we can define a predicate which is true of just those regions with denumerably infinite components. This would not be possible in a strictly 1st-order theory because of the Skolem-Löwenheim theorem.

8 CONCLUSION

We have specified a categorical theory of Region-Based Geometry that provides a secure and very general ontological foundation for representing qualitative spatial information. An application of the formalism to describing rigid body kinematics is explored in [2] and [3]. The 2nd-order nature of **RBG** poses severe problems for automated reasoning. However, our theory is not intended form the basis of a deductive mechanism; but rather to provide a logical framework with a precise semantics within which a variety of more practical representation languages (such as that of [17]) might be embedded.

Although our theory is extremely general it does have the limitation that it can only deal with a domain of entities having a given fixed dimension. For many applications it would be useful to be able to refer to entities of different dimensionality [11]. We would like to extend our theory to cater for this.

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