A General Recursive Schema for Argumentation Semantics

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Abstract. In argumentation theory, Dung's abstract framework provides a unifying view of several alternative semantics based on the notion of extension. Recently, a new semantics has been introduced to solve the problems related to counterintuitive results produced by literature proposals. In this semantics, an important role is played by a recursive schema in the definition of extensions. This paper proves that all the semantics encompassed by Dung's framework adhere to this property, not previously considered in the literature, which we call *SCC-recursiveness*. We argue that this notion plays a general role in the definition and computation of argumentation semantics.

1 INTRODUCTION

Argumentation theory is a framework for commonsense reasoning, which models the reasoning activity as the process of constructing and comparing arguments supporting conclusions. Since their construction proceeds by exploiting incomplete and uncertain information, arguments usually conflict, and this is modeled by an *attack relation* between them. As shown in [8], the variety of argumentation systems proposed in the literature differ along a number of dimensions, such as the language used to represent the arguments and the form of conflict between them. In particular, different underlying *argumentation semantics* introduce in a declarative way the criteria to determine, given a set of interacting arguments, which ones of them emerge as 'justified' from the conflict. Almost all of the argumentation semantics rely on the notion of *extension*, roughly consisting in a set of non-conflicting arguments: an argument is considered as justified if it belongs to all of the extensions. However, as pointed out in [8], two alternative approaches can be followed in this respect: in the so-called unique-status approach a single extension is always identified, while in the multiple-status approach several extensions may exist for a specific set of arguments. Moreover, specific proposals also differ in the form the underlying semantics is introduced. For instance, in [5, 9, 3] a fixed point definition is exploited, while in [6] the semantics is defined inductively by means of the notion of level.

A unifying framework, able to encompass a large variety of proposals, has been proposed by Dung in [5]. Here different semantics are introduced by means of fixed point definitions, all of them relying on the notion of *admissible set*. However, it has been shown in [1] that Dung's framework gives rise to counterintuitive results in some examples concerning cyclic attack relationships: to deal with them, a new semantics based on a *recursive* definition of extension has been introduced. In this paper, we show that also other existing semantics adhere to this recursive schema, which therefore appears to be a novel unifying concept underlying different argumentation theories.

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2 DUNG'S THEORY: A COUNTEREXAMPLE

The general theory proposed by Dung [5] is based on the primitive notion of *argumentation framework*:

Definition 1 *An argumentation framework is a pair* $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$, *where* $\mathcal{A}$ *is a set, and* $\rightarrow \subseteq (\mathcal{A} \times \mathcal{A})$ *is a binary relation on* $\mathcal{A}$.

The idea is that arguments are simply conceived as the elements of the set $\mathcal{A}$, whose origin is not specified, and the interaction between them is modeled by the binary relation of attack $\rightarrow$.

In the following, nodes that attack a given $\alpha \in \mathcal{A}$ are called *defeaters* of α, and form a set denoted as $\mathrm{parents}(\alpha)$:

Definition 2 *Given an argumentation framework* $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$ *and a node* $\alpha \in \mathcal{A}$, *we define* $\mathrm{parents}(\alpha) = \{\beta \in \mathcal{A} \mid \beta \rightarrow \alpha\}$. *If* $\mathrm{parents}(\alpha) = \emptyset$, *then* α *is called an* initial *node.*

Since we will frequently consider properties of sets of arguments, it is useful to extend to them the notations defined for the nodes:

Definition 3 *Given an argumentation framework* $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$, *a node* $\alpha \in \mathcal{A}$ *and two sets* $S, P \subseteq \mathcal{A}$, *we define:*

$$S \rightarrow \alpha \text{ iff } \exists \beta \in S : \beta \rightarrow \alpha$$
$$\alpha \rightarrow S \text{ iff } \exists \beta \in S : \alpha \rightarrow \beta$$
$$S \rightarrow P \text{ iff } \exists \alpha \in S, \beta \in P : \alpha \rightarrow \beta$$
$$\mathrm{parents}(S) = \{\alpha \in \mathcal{A} \mid \alpha \rightarrow S\}$$
$$\mathrm{outparents}(S) = \{\alpha \in \mathcal{A} \mid \alpha \notin S \land \alpha \rightarrow S\}$$

Acceptability and admissibility are the fundamental notions of Dung's theory:

Definition 4 *Given an argumentation framework* $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$:

- *A set* $S \subseteq \mathcal{A}$ *is* conflict-free *if and only if* $\nexists \alpha, \beta \in S$ *such that* $\alpha \rightarrow \beta$.
- *An argument* $\alpha \in \mathcal{A}$ *is* acceptable *with respect to a set* $S \subseteq \mathcal{A}$ *iff* $\forall \beta \in \mathcal{A}$, *if* $\beta \rightarrow \alpha$ *then also* $S \rightarrow \beta$.
- *A set* $S \subseteq \mathcal{A}$ *is* admissible *iff* S *is conflict-free and each argument in* S *is acceptable with respect to* S, *i.e.* $\forall \beta \in \mathcal{A}$ *such that* $\beta \rightarrow S$ *we have that* $S \rightarrow \beta$.

Building on the notion of admissible sets, Dung introduces three semantics, i.e. *complete*, *grounded*, and *preferred* semantics. The most satisfactory results are ensured by preferred semantics, which defines the extensions as maximal admissible sets. Limitations of preferred semantics have been pointed out in [1]: to have an idea, consider the argumentation frameworks shown in Figure 1, that differ only in the length of the leftmost cycle. It can be seen that AF_1

admits only one extension consisting of the node ϕ_2, while AF_2 admits three extensions whose intersection is empty. Therefore, ϕ_2 is justified in AF_1 while no argument is justified in AF_2. This turns out to be counterintuitive considering, for instance, that these argumentation frameworks can be regarded as simple variants of the example of conflicting witnesses proposed by Pollock in [7]. Here, the leftmost cycle represents a set of witnesses such that each of them questions the reliability of another one, and this *undercut* relation is arranged in a cycle. The two-length cycle on the right represents a couple of arguments with contradictory conclusions, known in the literature as *Nixon Diamond*, where ϕ_1 is based on an assertion of one of the witnesses. Intuitively, the length of the leftmost cycle, i.e. the number of witnesses, should be irrelevant. However, as described above, preferred semantics give a different treatment to ϕ_2 in the two cases.

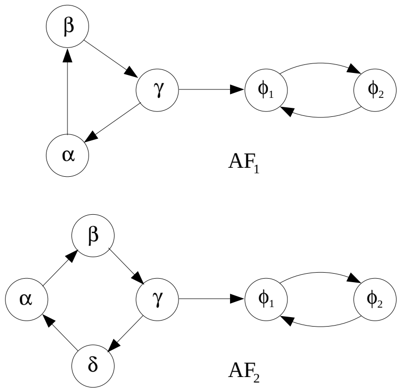

Figure 1. Different handling of cycles by preferred semantics

3 A NEW ARGUMENTATION SEMANTICS

To solve the problems described above, a new approach has been introduced in [1] which exploits the notion of strongly connected components of the argumentation framework:

Definition 5 *Given an argumentation framework* $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$, *two nodes* $\alpha, \beta \in \mathcal{A}$ *are* path-equivalent *iff either* $\alpha = \beta$ *or there is a path from* α *to* β *and a path from* β *to* α. *The strongly connected components of* AF *are the equivalence classes of vertices under the relation of path-equivalence. The set of the strongly connected components of* AF *is denoted as* $\mathrm{SCC}(\mathrm{AF})$.

Given a node $\alpha \in \mathcal{A}$, we will indicate the strongly connected component α belongs to as $\mathrm{SCC}(\alpha)$. We extend to strongly connected components the notion of defeaters, and we introduce the definition of *proper ancestors*:

Definition 6 *Given an argumentation framework* $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$ *and a strongly connected component* $S \in \mathrm{SCC}(\mathrm{AF})$, $\mathrm{sccparents}(S) = \{P \in \mathrm{SCC}(\mathrm{AF}) \mid P \neq S \wedge P \rightarrow S\}$, *and* $\mathrm{sccanc}(S) = \mathrm{sccparents}(S) \cup \bigcup_{P \in \mathrm{sccparents}(S)} \mathrm{sccanc}(P)$. *A strongly connected component* S *such that* $\mathrm{sccparents}(S) = \emptyset$ *is called* initial.

It is well-known that the graph obtained by considering strongly connected components as single nodes is acyclic: $\mathrm{sccanc}(S)$ include those strongly connected components that are antecedent to S in such a graph.

Definition 7 *Given an argumentation framework* $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$, *a set* $E \subseteq \mathcal{A}$ *and a strongly connected component* $S \in \mathrm{SCC}(\mathrm{AF})$, *we define:*

- $S^D(E) = \{\alpha \in S \mid (E \cap \mathrm{outparents}(S)) \rightarrow \alpha\}$
- $S^P(E) = \{\alpha \in S \mid (E \cap \mathrm{outparents}(S)) \not\rightarrow \alpha \wedge \exists \beta \in (\mathrm{outparents}(S) \cap \mathrm{parents}(\alpha)) : E \not\rightarrow \beta\}$
- $S^U(E) = S \setminus (S^D(E) \cup S^P(E)) =$
  $= \{\alpha \in S \mid (E \cap \mathrm{outparents}(S)) \not\rightarrow \alpha \wedge \forall \beta \in (\mathrm{outparents}(S) \cap \mathrm{parents}(\alpha))\ E \rightarrow \beta\}$

In words, if the set E is an extension, the set $S^D(E)$ consists of the nodes of S attacked by E from outside S, the set $S^U(E)$ consists of the nodes of S that are not attacked by E and are defended by E (i.e. their defeaters from outside S are all attacked by E), and $S^P(E)$ consists of the nodes of S that are not attacked nor defended by E. It is easy to see that $S^D(E)$, $S^P(E)$ and $S^U(E)$ are determined only by the elements of E that belong to the strongly connected components in $\mathrm{sccanc}(S)$.

Finally, we need the notion of *restriction* of an argumentation framework to a given subset of its nodes:

Definition 8 *Let* $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$ *be an argumentation framework, and let* S *be a set* $S \subseteq \mathcal{A}$. *The* restriction *of* AF *to* S *is the argumentation framework* $\mathrm{AF}{\downarrow_S} = < S, \rightarrow \cap (S \times S) >$.

The semantics introduced in [1] uses a definition of extension that can be formulated as follows:

Definition 9 *Given an argumentation framework* $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$, *a set* $E \subseteq \mathcal{A}$ *is an extension, denoted as* $E \in \mathcal{RE}_{\mathrm{AF}}$, *iff*

- $E \in \mathcal{MI}_{\mathrm{AF}}$ *if* $|\mathrm{SCC}(\mathrm{AF})| = 1$
- $\forall S \in \mathrm{SCC}(\mathrm{AF})\ (E \cap S) \in \mathcal{RE}_{\mathrm{AF}\downarrow_{(S^P(E) \cup S^U(E))}}$ *otherwise*

where $\mathcal{MI}_{\mathrm{AF}}$ *denotes the set of maximal conflict-free sets of* AF.

It can be seen that this semantics overcomes the limitations of preferred semantics concerning odd-length cycles, in particular it ensures an equal treatment of the argumentation frameworks presented in Figure 1. Actually, this is obtained by replacing the admissibility requirement with the less demanding notion of maximal conflict-free set, arranged in a recursive schema, which is able to properly constrain the set of extensions prescribed by the semantics. In the next section, we show that this schema is not specific to our proposal, but underlies all semantics encompassed by Dung's framework.

4 ARGUMENTATION SEMANTICS: A RECURSIVE CHARACTERIZATION

4.1 Extending Dung's theory

In order to accomplish our analysis, it is necessary to generalize Dung's theory by considering a specific subset C of $\mathcal{A}$ from which acceptable arguments (that compose the extensions) are selected. Accordingly, given an argumentation framework $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$ and a set $C \subseteq \mathcal{A}$, we consider admissible sets that are included in C:

Definition 10 *Given an argumentation framework* $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$ *and a set* $C \subseteq \mathcal{A}$, *we define* $\mathcal{AS}_{\mathrm{AF}}(C) = \{E \subseteq C \mid E \textit{ is admissible}\}$.

The notion of complete extension is introduced in [5] as a unifying concept underlying various existing semantics. We directly introduce the relevant definition in the context of the generalized framework.

Definition 11 *Given an argumentation framework* $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$ *and a set* $C \subseteq \mathcal{A}$*, a set* $S \subseteq \mathcal{A}$ *is a* complete extension in C *iff* $S \in \mathcal{AS}_{\mathrm{AF}}(C)$*, and every argument* $\alpha \in C$ *which is acceptable with respect to* S *belongs to* S*. The set of complete extensions in* C *will be denoted as* $\mathcal{CE}_{\mathrm{AF}}(C)$*.*

In the generalized framework, the notion of preferred extension is introduced as follows:

Definition 12 *Given an argumentation framework* $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$*, let us consider a set of arguments* $C \subseteq \mathcal{A}$*. A* preferred extension in C *is a maximal element (with respect to set inclusion) of* $\mathcal{AS}_{\mathrm{AF}}(C)$*. The set of preferred extensions in* C *will be denoted as* $\mathcal{PE}_{\mathrm{AF}}(C)$*.*

In other terms, $E \in \mathcal{PE}_{\mathrm{AF}}(C)$ if and only if E is a maximal set such that $E \subseteq C$ and E is admissible.

A relevant question concerns the existence of a preferred extension for any argumentation framework AF and for all sets $C \subseteq \mathcal{A}$. In this respect, we are able to extend Dung's results:

Theorem 1 *Given an argumentation framework* $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$ *and a set* $C \subseteq \mathcal{A}$*:*

- *The elements of* $\mathcal{AS}_{\mathrm{AF}}(C)$*, i.e. the admissible subsets of* C*, form a complete partial order.*
- *For all* $F \in \mathcal{AS}_{\mathrm{AF}}(C)$*, there is* $E \in \mathcal{PE}_{\mathrm{AF}}(C)$ *such that* $F \subseteq E$*.*

Corollary 1 *Given an argumentation framework* $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$ *and a set* $C \subseteq \mathcal{A}$*,* $\mathcal{PE}_{\mathrm{AF}}(C)$ *is non empty, i.e. there is always a preferred extension* $E \in \mathcal{PE}_{\mathrm{AF}}(C)$*.*

The grounded semantics, introduced by Pollock in [6], is probably the most representative proposal in the context of the unique-status approach. In our framework, it can be defined in terms of the least fixed point of the characteristic function.

Definition 13 *With reference to an argumentation framework* $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$ *and a set of arguments* $C \subseteq \mathcal{A}$*, the* characteristic function of AF in C*, denoted as* $\mathrm{F}_{\mathrm{AF},C}$*, is defined as follows:*

$$\mathrm{F}_{\mathrm{AF},C} : 2^C \rightarrow 2^C$$
$$\mathrm{F}_{\mathrm{AF},C}(Q) = \{\alpha \mid \alpha \in C,\ \alpha \textit{ acceptable with respect to } Q\}$$

It is easy to see that $\mathrm{F}_{\mathrm{AF},C}$ is monotonic (with respect to $\subseteq$).

Definition 14 *Given an argumentation framework* $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$ *and a set* $C \subseteq \mathcal{A}$*, the* grounded extension of AF in C*, denoted as* $\mathrm{GE}_{\mathrm{AF}}(C)$*, is the least (with respect to* $\subseteq$*) fixed point of* $\mathrm{F}_{\mathrm{AF},C}$*.*

Notice that by definition $\mathrm{GE}_{\mathrm{AF}}(C) \subseteq C$. Also in this case we have a positive result concerning the existence of the grounded extension:

Lemma 1 *For any argumentation framework* $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$ *and for all sets* $C \subseteq \mathcal{A}$*,* $\mathrm{GE}_{\mathrm{AF}}(C)$ *exists and is unique.*

Finally, the following relation between grounded and complete extensions can be drawn:

Proposition 1 *Given an argumentation framework* $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$ *and a set of arguments* $C \subseteq \mathcal{A}$*, we have that* $\mathrm{GE}_{\mathrm{AF}}(C)$ *is the least (with respect to set inclusion) complete extension in* C *(i.e. the least element in* $\mathcal{CE}_{\mathrm{AF}}(C)$*).*

Dung's original definitions are recovered by letting $C = \mathcal{A}$. Therefore, a recursive formulation of the extended definitions also covers the original ones. In the following subsections, we show that both admissible sets and all kinds of extensions can be recursively characterized along strongly connected components. These basic results will be exploited to obtain the general recursive schema finally presented in Section 5.

4.2 Admissible sets

The characterization of admissible sets with respect to strongly connected components is achieved by Proposition 2, which requires two preliminary lemmas (proofs are omitted due to space limitations).

Lemma 2 *Given an argumentation framework* $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$*, let* $E \subseteq \mathcal{A}$ *be an admissible set in* AF*, and* $\alpha \in \mathcal{A}$ *be an argument acceptable with respect to* E*. Denoting* $\mathrm{SCC}(\alpha)$ *as* S*, we have that:*

- $\alpha \in S^U(E)$*; and*
- *in the argumentation framework* $\mathrm{AF}{\downarrow}_{(S^P(E) \cup S^U(E))}$*,* α *is acceptable with respect to* $(E \cap S)$*.*

Lemma 3 *Given an argumentation framework* $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$*, let* $E \subseteq \mathcal{A}$ *be a set of arguments such that,* $\forall S \in \mathrm{SCC}(\mathrm{AF})$

$$(E \cap S) \in \mathcal{AS}_{\mathrm{AF}{\downarrow}_{(S^P(E) \cup S^U(E))}}(S^U(E))$$

Given $\hat{S} \in \mathrm{SCC}(\mathrm{AF})$*, if* $\alpha \in \hat{S}^U(E)$ *is an argument acceptable with respect to* $(E \cap \hat{S})$ *in the argumentation framework* $\mathrm{AF}{\downarrow}_{(\hat{S}^P(E) \cup \hat{S}^U(E))}$*, then* α *is acceptable with respect to* E *in* AF*.*

Proposition 2 *Given an argumentation framework* $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$*, let us consider a set of arguments* $E \subseteq \mathcal{A}$*. Then,* $\forall C \subseteq \mathcal{A}$*,* $E \in \mathcal{AS}_{\mathrm{AF}}(C)$ *if and only if* $\forall S \in \mathrm{SCC}(\mathrm{AF})$

$$(E \cap S) \in \mathcal{AS}_{\mathrm{AF}{\downarrow}_{(S^P(E) \cup S^U(E))}}(S^U(E) \cap C)$$

Proof: First, let us prove that if E is admissible then it satisfies the conditions relevant to a generic strongly connected component $S \in \mathrm{SCC}(\mathrm{AF})$. According to the definition of $\mathcal{AS}_{\mathrm{AF}}(C)$, $E \subseteq C$ and $\forall \alpha \in E$, α is acceptable with respect to E. As a consequence, on the basis of Lemma 2 we have that $\forall \alpha \in (E \cap S)$, $\alpha \in S^U(E)$, therefore $(E \cap S) \subseteq (S^U(E) \cap C)$. Moreover, by the same lemma α is acceptable with respect to $(E \cap S)$ in the argumentation framework $\mathrm{AF}{\downarrow}_{(S^P(E) \cup S^U(E))}$. This, as well as the fact that E is admissible and therefore conflict-free, entails that $(E \cap S)$ is admissible in the argumentation framework $\mathrm{AF}{\downarrow}_{(S^P(E) \cup S^U(E))}$, and therefore that $(E \cap S) \in \mathcal{AS}_{\mathrm{AF}{\downarrow}_{(S^P(E) \cup S^U(E))}}(S^U(E) \cap C)$.
As far as the other direction of the proof is concerned, we first notice that, by the hypothesis, $\forall S \in \mathrm{SCC}(\mathrm{AF})$ $(E \cap S) \subseteq (S^U(E) \cap C) \subseteq (S \cap C)$, therefore $E \subseteq C$: in order to prove the claim, we have only to show that E is admissible in AF.
Let us first show that E is conflict-free by reasoning by contradiction, i.e. let us suppose that $\exists \alpha, \beta \in E : \beta \rightarrow \alpha$. Let us denote $\mathrm{SCC}(\alpha)$ (in AF) as S. Clearly, it cannot be the case that $\mathrm{SCC}(\alpha) = \mathrm{SCC}(\beta)$, since in this case $(E \cap S)$ would not be conflict-free, thus contradicting the hypothesis concerning its admissibility in $\mathrm{AF}{\downarrow}_{(S^P(E) \cup S^U(E))}$. As a consequence, $\beta \in (E \cap \mathrm{outparents}(S))$, therefore $\alpha \in S^D(E)$ by the definition of $S^D(E)$. However, this contradicts the fact that $\alpha \in (E \cap S)$, which according to the hypothesis is contained in $S^U(E)$.
In order to complete the proof, we have to prove that a generic $\alpha \in E$ is acceptable with respect to E. If we denote $\mathrm{SCC}(\alpha)$ (in AF) as S, we have that $\alpha \in (E \cap S)$, and by the hypothesis $(E \cap S) \in \mathcal{AS}_{\mathrm{AF}{\downarrow}_{(S^P(E) \cup S^U(E))}}(S^U(E) \cap C)$. Therefore, $\alpha \in S^U(E)$, and α is acceptable with respect to $(E \cap S)$ in $\mathrm{AF}{\downarrow}_{(S^P(E) \cup S^U(E))}$. Since the hypothesis entails that $\forall S \in \mathrm{SCC}(\mathrm{AF})$ $(E \cap S) \in \mathcal{AS}_{\mathrm{AF}{\downarrow}_{(S^P(E) \cup S^U(E))}}(S^U(E))$, Lemma 3 can be applied to α, entailing that α is acceptable with respect to E (in AF). □

4.3 Complete semantics

The following proposition shows that also complete extensions are in correspondence with a recursive decomposition along strongly connected components.

Proposition 3 *Given an argumentation framework* $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$, *let us consider a sets of arguments* $E \subseteq \mathcal{A}$. *Then,* $\forall C \subseteq \mathcal{A}$, $E \in \mathcal{CE}_{\mathrm{AF}}(C)$ *if and only if* $\forall S \in \mathrm{SCC}(\mathrm{AF})$

$$(E \cap S) \in \mathcal{CE}_{\mathrm{AF}\downarrow_{(S^P(E) \cup S^U(E))}}(S^U(E) \cap C)$$

Proof: As for the first direction of the proof, if $E \in \mathcal{CE}_{\mathrm{AF}}(C)$ then in particular $E \in \mathcal{AS}_{\mathrm{AF}}(C)$, therefore Proposition 2 entails that

$$\forall S \in \mathrm{SCC}(\mathrm{AF})\ (E \cap S) \in \mathcal{AS}_{\mathrm{AF}\downarrow_{(S^P(E) \cup S^U(E))}}(S^U(E) \cap C) \tag{1}$$

As a consequence, we have only to show that $\forall \alpha \in (S^U(E) \cap C)$ such that α is acceptable with respect to $(E \cap S)$ in $\mathrm{AF}\downarrow_{(S^P(E) \cup S^U(E))}$, $\alpha \in (E \cap S)$. First, we notice that Lemma 3 can be applied to α, since (1) entails that $\forall S \in \mathrm{SCC}(\mathrm{AF})\ (E \cap S) \in \mathcal{AS}_{\mathrm{AF}\downarrow_{(S^P(E) \cup S^U(E))}}(S^U(E))$. On the basis of this lemma, α is acceptable with respect to E (in AF). Moreover, $\alpha \in (S^U(E) \cap C)$, therefore in particular $\alpha \in C$. As a consequence, from the hypothesis that $E \in \mathcal{CE}_{\mathrm{AF}}(C)$ it follows that $\alpha \in E$ and therefore $\alpha \in (E \cap S)$.
As for the other direction of the proof, according to Definition 11 we have that $\forall S \in \mathrm{SCC}(\mathrm{AF})$ the following conditions hold:

$$(E \cap S) \in \mathcal{AS}_{\mathrm{AF}\downarrow_{(S^P(E) \cup S^U(E))}}(S^U(E) \cap C) \tag{2}$$

$$\forall \alpha \in (S^U(E) \cap C) : \alpha \text{ acceptable with respect to } (E \cap S) \text{ in } \mathrm{AF}\downarrow_{(S^P(E) \cup S^U(E))},\ \alpha \in (E \cap S) \tag{3}$$

Thus, on the basis of (2) Proposition 2 entails that $E \in \mathcal{AS}_{\mathrm{AF}}(C)$, therefore we have only to prove that $\forall \alpha \in C$ such that α is acceptable with respect to E, $\alpha \in E$. Denoting $\mathrm{SCC}(\alpha)$ as S (where $S \in \mathrm{SCC}(\mathrm{AF})$), on the basis of Lemma 2 we have that $\alpha \in S^U(E)$, so that $\alpha \in (S^U(E) \cap C)$, and α is acceptable with respect to $(E \cap S)$ in $\mathrm{AF}\downarrow_{(S^P(E) \cup S^U(E))}$. Then, taking into account (3) we have that $\alpha \in (E \cap S)$, therefore $\alpha \in E$. □

4.4 Preferred semantics

Also preferred extensions fit the decomposition schema along strongly connected components, as shown by Proposition 4, based on the following lemma (proof is omitted due to space limitations).

Lemma 4 *Given an argumentation framework* $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$, *let* $E \subseteq \mathcal{A}$ *be an admissible set in* AF *and let* $S \in \mathrm{SCC}(\mathrm{AF})$. *Let* $\hat{E}$ *be a set of arguments such that* $(E \cap S) \subseteq \hat{E} \subseteq S^U(E)$, *and* $\hat{E}$ *is admissible in the argumentation framework* $\mathrm{AF}\downarrow_{(S^P(E) \cup S^U(E))}$. *Then, we have that* $(E \cup \hat{E})$ *is admissible in* AF.

Proposition 4 *Given an argumentation framework* $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$, *let us consider a set of arguments* $E \subseteq \mathcal{A}$. *Then,* $\forall C \subseteq \mathcal{A}$, $E \in \mathcal{PE}_{\mathrm{AF}}(C)$ *if and only if* $\forall S \in \mathrm{SCC}(\mathrm{AF})$

$$(E \cap S) \in \mathcal{PE}_{\mathrm{AF}\downarrow_{(S^P(E) \cup S^U(E))}}(S^U(E) \cap C)$$

Proof: As far as the first direction of the proof is concerned, let us assume that $E \in \mathcal{PE}_{\mathrm{AF}}(C)$. By definition, $E \in \mathcal{AS}_{\mathrm{AF}}(C)$, therefore, on the basis of Proposition 2, we have that $\forall S \in \mathrm{SCC}(\mathrm{AF})$

$$(E \cap S) \in \mathcal{AS}_{\mathrm{AF}\downarrow_{(S^P(E) \cup S^U(E))}}(S^U(E) \cap C)$$

Let us reason by contradiction, assuming that $\exists \hat{S} \in \mathrm{SCC}(\mathrm{AF})$ such that $(E \cap \hat{S})$ is not maximal among the sets included in $\mathcal{AS}_{\mathrm{AF}\downarrow_{(\hat{S}^P(E) \cup \hat{S}^U(E))}}(\hat{S}^U(E) \cap C)$. According to Theorem 1, there must be a set $\hat{E}$ such that

- $(E \cap \hat{S}) \subset \hat{E} \subseteq (\hat{S}^U(E) \cap C)$, and
- $\hat{E} \in \mathcal{AS}_{\mathrm{AF}\downarrow_{(\hat{S}^P(E) \cup \hat{S}^U(E))}}(\hat{S}^U(E) \cap C)$.

Taking into account that E is admissible in AF, Lemma 4 entails that the set $E' \triangleq E \cup \hat{E}$ is admissible in AF. However, it is easy to see that E is strictly contained in E' and that $E' \subseteq C$, contradicting the maximality of E among the admissible sets of AF included in C.
Let us turn now to the other direction of the proof, assuming that $\forall S \in \mathrm{SCC}(\mathrm{AF})\ (E \cap S) \in \mathcal{PE}_{\mathrm{AF}\downarrow_{(S^P(E) \cup S^U(E))}}(S^U(E) \cap C)$. On the basis of Proposition 2, $E \in \mathcal{AS}_{\mathrm{AF}}(C)$: in order to prove that E is also a preferred extension, we reason again by contradiction, supposing that $\exists E' \subseteq C, E \subset E' : E' \in \mathcal{PE}_{\mathrm{AF}}(C)$ (notice that E' exists by Theorem 1). Since $E \subset E'$, there must be at least a strongly connected component $S \in \mathrm{SCC}(\mathrm{AF})$ such that $(E \cap S) \subset (E' \cap S)$: taking into account the acyclicity of the strongly connected components, there exists in particular $\hat{S} \in \mathrm{SCC}(\mathrm{AF})$ such that

$$\forall S \in \mathrm{SCC}(\mathrm{AF}) : S \in \mathrm{sccanc}(\hat{S}),\ (E' \cap S) = (E \cap S) \tag{4}$$

$$(E \cap \hat{S}) \subset (E' \cap \hat{S}) \tag{5}$$

Since $E' \in \mathcal{AS}_{\mathrm{AF}}(C)$, Proposition 2 entails that $(E' \cap \hat{S}) \in \mathcal{AS}_{\mathrm{AF}\downarrow_{(\hat{S}^P(E') \cup \hat{S}^U(E'))}}(\hat{S}^U(E') \cap C)$. Taking into account (4), it is easy to see that $\hat{S}^U(E') = \hat{S}^U(E)$ and $\hat{S}^P(E') = \hat{S}^P(E)$, therefore $(E' \cap \hat{S}) \in \mathcal{AS}_{\mathrm{AF}\downarrow_{(\hat{S}^P(E) \cup \hat{S}^U(E))}}(\hat{S}^U(E) \cap C)$. However, on the basis of (5) we have that $(E \cap \hat{S}) \subset (E' \cap \hat{S})$, contradicting the hypothesis that $(E \cap \hat{S}) \in \mathcal{PE}_{\mathrm{AF}\downarrow_{(\hat{S}^P(E) \cup \hat{S}^U(E))}}(\hat{S}^U(E) \cap C)$. □

4.5 Grounded semantics

In this section, we prove that the decomposition schema also holds for grounded semantics.

Proposition 5 *Given an argumentation framework* $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$, *let us consider a set of arguments* $E \subseteq \mathcal{A}$. *Then,* $\forall C \subseteq \mathcal{A}$, $E = \mathrm{GE}_{\mathrm{AF}}(C)$ *if and only if* $\forall S \in \mathrm{SCC}(\mathrm{AF})$

$$(E \cap S) = \mathrm{GE}_{\mathrm{AF}\downarrow_{(S^P(E) \cup S^U(E))}}(S^U(E) \cap C)$$

Proof: Let us consider the first part of the proof, by supposing that $E = \mathrm{GE}_{\mathrm{AF}}(C)$. On the basis of Proposition 1, E is in particular a complete extension in C, i.e. $E \in \mathcal{CE}_{\mathrm{AF}}(C)$, therefore Proposition 3 entails that $\forall S \in \mathrm{SCC}(\mathrm{AF})\ (E \cap S) \in \mathcal{CE}_{\mathrm{AF}\downarrow_{(S^P(E) \cup S^U(E))}}(S^U(E) \cap C)$. Taking into account Proposition 1, we have to prove that $\forall S \in \mathrm{SCC}(\mathrm{AF})\ (E \cap S)$ is the least element (with respect to set inclusion) in $\mathcal{CE}_{\mathrm{AF}\downarrow_{(S^P(E) \cup S^U(E))}}(S^U(E) \cap C)$. We reason by contradiction, supposing that there is at least one strongly connected component where the thesis is not verified. In particular, since the strongly connected components of AF make up an acyclic graph, we can choose $\hat{S} \in \mathrm{SCC}(\mathrm{AF})$ such that:

- $\forall S \in \mathrm{SCC}(\mathrm{AF}) : S \in \mathrm{sccanc}(\hat{S}),\ (E \cap S) = \mathrm{GE}_{\mathrm{AF}\downarrow_{(S^P(E) \cup S^U(E))}}(S^U(E) \cap C)$; and
- $\exists \hat{E} \subset (E \cap \hat{S}), \hat{E} = \mathrm{GE}_{\mathrm{AF}\downarrow_{(\hat{S}^P(E) \cup \hat{S}^U(E))}}(\hat{S}^U(E) \cap C)$.

Note that in case $\hat{S}$ is initial, the first condition is trivially verified. Moreover, the second condition follows from the fact that, on the basis of Lemma 1, $\mathrm{GE}_{\mathrm{AF}\downarrow_{(\hat{S}^P(E)\cup\hat{S}^U(E))}}(\hat{S}^U(E)\cap C)$ must exist, and according to Proposition 1 it is included in all the elements of $\mathcal{CE}_{\mathrm{AF}\downarrow_{(\hat{S}^P(E)\cup\hat{S}^U(E))}}(\hat{S}^U(E)\cap C)$.
Now, taking again into account that the strongly connected components of AF make up an acyclic graph, it is easy to see that it is possible to construct a set E' such that:

- $\forall S \in \mathrm{SCC(AF)} : S \in \mathrm{sccanc}(\hat{S}),\ (E' \cap S) = (E \cap S)$;
- $(E' \cap \hat{S}) = \hat{E}$;
- $\forall S \in \mathrm{SCC(AF)}\ (E' \cap S) = \mathrm{GE}_{\mathrm{AF}\downarrow_{(S^P(E')\cup S^U(E'))}}(S^U(E') \cap C)$

To this purpose, it is obviously possible to construct a set E'_* which satisfies the first two conditions concerning any strongly connected component $S \in (\hat{S} \cup \mathrm{sccanc}(\hat{S})$. Thus, it turns out that $S^U(E'_*) = S^U(E)$ and $S^P(E'_*) = S^P(E)$, and, as a consequence, E'_* satisfies the third condition too for any such S (taking into account the properties of E and $\hat{E}$ stated above). Now, E' can be obtained constructively from E'_* by proceeding along the other strongly connected components of the defeat graph: in fact $\forall S \in \mathrm{SCC(AF)}$ $\mathrm{GE}_{\mathrm{AF}\downarrow_{(S^P(E')\cup S^U(E'))}}(S^U(E') \cap C)$ always exists by Lemma 1.
Now, by Proposition 1, we have that $\forall S \in \mathrm{SCC(AF)}\ (E' \cap S) \in \mathcal{CE}_{\mathrm{AF}\downarrow_{(S^P(E')\cup S^U(E'))}}(S^U(E') \cap C)$. As a consequence, on the basis of Proposition 3 $E' \in \mathcal{CE}_{\mathrm{AF}}(C)$, while since $(E' \cap \hat{S}) = \hat{E} \subset (E \cap \hat{S})$ it is not true that $E \subseteq E'$. However, this contradicts the hypothesis that $E = \mathrm{GE_{AF}}(C)$, and as such the least complete extension in C of AF (see Proposition 1).
Let us turn now to the other direction of the proof, by supposing that $\forall S \in \mathrm{SCC(AF)}\ (E \cap S) = \mathrm{GE}_{\mathrm{AF}\downarrow_{(S^P(E)\cup S^U(E))}}(S^U(E) \cap C)$. On the basis of Proposition 1, we have that $\forall S \in \mathrm{SCC(AF)}(E \cap S) \in \mathcal{CE}_{\mathrm{AF}\downarrow_{(S^P(E)\cup S^U(E))}}(S^U(E) \cap C)$, therefore Proposition 3 entails that $E \in \mathcal{CE}_{\mathrm{AF}}(C)$. As a consequence, taking into account Proposition 1 we have only to prove that E is the least element of $\mathcal{CE}_{\mathrm{AF}}(C)$. We reason by contradiction, assuming that the grounded extension $E' = \mathrm{GE_{AF}}(C)$, which must exist by Lemma 1 and is a subset of E by Proposition 1, is strictly included in E. Thus, there must be at least a strongly connected component S such that $(E' \cap S) \subset (E \cap S)$: since the strongly connected components form an acyclic graph, there is in particular a strongly connected component $\hat{S}$ such that:

$$\forall S \in \mathrm{SCC(AF)} : S \in \mathrm{sccanc}(\hat{S}),\ (E' \cap S) = (E \cap S) \quad (6)$$

$$(E' \cap \hat{S}) \subset (E \cap \hat{S}) \quad (7)$$

Moreover, since $E' = \mathrm{GE_{AF}}(C) \in \mathcal{CE}_{\mathrm{AF}}(C)$, on the basis of Proposition 3 applied to $\hat{S}$ it must be the case that $(E' \cap \hat{S}) \in \mathcal{CE}_{\mathrm{AF}\downarrow_{(\hat{S}^P(E')\cup\hat{S}^U(E'))}}(\hat{S}^U(E') \cap C)$. Taking into account (6), it is easy to see that $\hat{S}^U(E') = \hat{S}^U(E)$ and $\hat{S}^P(E') = \hat{S}^P(E)$, therefore $(E' \cap \hat{S}) \in \mathcal{CE}_{\mathrm{AF}\downarrow_{(\hat{S}^P(E)\cup\hat{S}^U(E))}}(\hat{S}^U(E) \cap C)$. However, according to (7) we have that $(E' \cap \hat{S})$ is strictly included in $(E \cap \hat{S})$, contradicting the hypothesis (referred to $\hat{S}$) that $(E \cap \hat{S}) = \mathrm{GE}_{\mathrm{AF}\downarrow_{(\hat{S}^P(E)\cup\hat{S}^U(E))}}(\hat{S}^U(E) \cap C)$ and therefore, on the basis of Proposition 1, that $(E \cap \hat{S})$ is the least element of $\mathcal{CE}_{\mathrm{AF}\downarrow_{(\hat{S}^P(E)\cup\hat{S}^U(E))}}(\hat{S}^U(E) \cap C)$. □

5 A GENERAL SCHEMA FOR ARGUMENTATION

The above results suggest to introduce a new fundamental concept for argumentation semantics, called *SCC-recursiveness*, that characterizes all the considered semantics:

Definition 15 *A given argumentation semantics is* SCC-recursive *if, with reference to a generic argumentation framework* $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$*, a set* $E \subseteq \mathcal{A}$ *is an extension iff* $E \in \mathcal{FG}_{\mathrm{AF}}(\mathcal{A})$*,*
where, for all sets $C \subseteq \mathcal{A}$*,* $E \in \mathcal{FG}_{\mathrm{AF}}(C)$ *iff*

- $E \in \mathcal{FG}^*_{\mathrm{AF}}(C)$ *if* $|\mathrm{SCC(AF)}| = 1$
- $\forall S \in \mathrm{SCC(AF)}\ (E \cap S) \in \mathcal{FG}_{\mathrm{AF}\downarrow_{(S^P(E)\cup S^U(E))}}(S^U(E) \cap C)$ *otherwise*

where $\mathcal{FG}^*_{\mathrm{AF}}(C)$ *is a function that, given an argumentation framework* $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$ *such that* $|\mathrm{SCC(AF)}| = 1$ *and a set* $C \subseteq \mathcal{A}$*, gives a subset of* $2^{\mathcal{A}}$*.*

The function $\mathcal{FG}^*_{\mathrm{AF}}(C)$, which we call *base function*, returns the extensions of a generic argumentation framework with a unique strongly connected component. Since a particular SCC-recursive semantics is identified by its own base function, it is interesting to notice that to define an argumentation semantics it is sufficient to specify its behavior only on single-SCC argumentation frameworks. In particular, for traditional complete, preferred and grounded semantics the base function $\mathcal{FG}^*_{\mathrm{AF}}(C)$ turns out to be $\mathcal{CE}_{\mathrm{AF}}(C)$, $\mathcal{PE}_{\mathrm{AF}}(C)$ and $\mathrm{GE_{AF}}(C)$, respectively, while for the new semantics introduced in [1] $\mathcal{FG}^*_{\mathrm{AF}}(C)$ is $\mathcal{MI}_{\mathrm{AF}}$.

The importance of this result is twofold. First, it supports the development of efficient and incremental algorithms based on local computations at the level of strongly connected components. In particular, it is reasonable to suppose that a significant gain would be obtained by developing an SCC-based variant of the backtracking algorithms to compute the extensions proposed in [4]. More significantly, the family of SCC-recursive semantics appears to be a very general framework which eases the investigations of further argumentation semantics exploiting alternative definitions of the base function (see [2]). Both directions will be explored in future work.

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