A Spatial Logic of Betweenness

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Abstract. In this paper, I present a region-based spatial logic of the ternary betweenness relation. I provide semantics and a complete axiomatization of this first-order theory.

1 INTRODUCTION

In this paper, I present a first-order spatial logic of the region-based betweenness relation. There are certain types of calculi [1, 2, 3] in qualitative spatial reasoning literature which base their formalisms on different primitives, aimed at solving different reprsentation problems of the spatial domain. I investigate the important propeties of the logic of the betweenness relation, a much stronger primitive than that of connection [1, 3].

I present semantics and a complete first-order axiomatization of region-based betweenness ($\mathcal{B}$ for short) from which we show that our first-order theory is sound and complete. Note that the sort of completeness we consider is in the weak sense which means that every true formulae is provable in contrast to absolute sense, where every formulae of the theory is either true or false. The completeness proof is inspired by the application of the Henkin method in [1].

The structure of the rest of the paper is as follows: Section 2 gives the foundations of the first-order theory $\mathcal{B}$. Section 3 is devoted to the soundness and completeness $\mathcal{B}$ based on the structures presented in Section 2. Finally in Section 4 we have a glance at the future work.

2 THEORY OF BETWEENNESS

The 1st order language of betweenness, $\mathcal{L}$, contains a ternary primitive relation β, denumerably-infinite number of variables (p, q, etc.) and constants (a, b, etc.). $\beta(a, b, c)$ is read as 'region a occurs in between of regions b and c' The relation symbols are sometimes "overloaded" by using the same symbol with different number of arguments. For example it should be understood that NTP(x, y) is: 'x is a Non Tangential Part of y', where as NTP(x, y, z) is: 'x is a Non Tangential Part of the betweenness of y and z'. The formal theory of $\mathcal{B}$ is given as follows:

(A1) $\forall xyz[\beta(x, y, z) \rightarrow \beta(x, z, y)]$
(A2) $\forall xy[\forall zw[\beta(x, z, w) \leftrightarrow \beta(y, z, w)] \rightarrow x = y]$

(A1) is the symmetry axiom for the last two arguments of β and (A2) is the identity axiom.

(D1) $\mathrm{P}(x, y) \equiv_d \forall zw[\beta(x, z, w) \rightarrow \beta(y, z, w)]$
(D2) $\mathrm{O}(x, y) \equiv_d \exists z[\mathrm{P}(z, x) \wedge \mathrm{P}(z, y)]$
(D3) $\mathrm{P}(x, y, z) \equiv_d \forall w[\mathrm{P}(w, x) \rightarrow \beta(w, y, z)]$
(D4) $\mathrm{O}(x, y, z) \equiv_d \exists w[\mathrm{P}(w, x) \wedge \mathrm{P}(w, y, z)]$
(D5) $\mathrm{EC}(x, y, z) \equiv_d \beta(x, y, z) \wedge \neg \mathrm{O}(x, y, z)$
(D6) $\mathrm{DC}(x, y, z) \equiv_d \neg \beta(x, y, z)$
(D7) $\mathrm{BL}(x, y, z) \equiv_d \forall w[\neg \mathrm{O}(w, x) \wedge \mathrm{P}(w, y, z) \rightarrow [\mathrm{P}(w, x, y) \vee \mathrm{P}(w, x, z)]]$
(D8) $\mathrm{TP}(x, y, z) \equiv_d \mathrm{P}(x, y, z) \wedge \neg \mathrm{BL}(x, y, z) \wedge \exists wq[\mathrm{EC}(x, w, q) \wedge \neg \exists p[\mathrm{O}(p, y, z) \wedge \mathrm{O}(p, w, q)]]$
(D9) $\mathrm{NTP}(x, y, z) \equiv_d \mathrm{P}(x, y, z) \wedge \forall pq[\beta(x, p, q) \rightarrow \exists w[\mathrm{P}(w, y, z) \wedge \mathrm{P}(w, p, q)]]$
(D10) $\mathrm{EC}(x, y) \equiv_d \forall zw[[\mathrm{TP}(x, z, w) \wedge \neg \mathrm{O}(y, z, w)] \rightarrow \mathrm{EC}(y, z, w)]$
(D11) $\mathrm{TP}(x, y) \equiv_d \mathrm{P}(x, y) \wedge \exists z[\mathrm{EC}(x, z) \wedge \mathrm{EC}(y, z)]$
(D12) $\mathrm{NTP}(x, y) \equiv_d \mathrm{P}(x, y) \wedge \neg \exists z[\mathrm{EC}(x, z) \wedge \mathrm{EC}(y, z)]$
(D13) $\mathrm{C}(x, y) \equiv_d \mathrm{O}(x, y) \vee \mathrm{EC}(x, y)$
(D14) $\mathrm{DC}(x, y) \equiv_d \neg \mathrm{C}(x, y)$
(D15) $\mathrm{PO}(x, y, z) \equiv_d \mathrm{O}(x, y, z) \wedge \neg \mathrm{P}(x, y, z) \wedge \neg \mathrm{BL}(x, y, z)$
(D16) $\mathrm{TBL}(x, y, z) \equiv_d \mathrm{BL}(x, y, z) \wedge \mathrm{P}(x, y, z)$
(D17) $\mathrm{OBL}(x, y, z) \equiv_d \mathrm{BL}(x, y, z) \wedge \mathrm{O}(x, y, z) \wedge \neg \mathrm{P}(x, y, z)$

The definitions (D1)-(D17) are standard except for relations BL(x,y,z), TBL(x,y,z) and OBL(x,y,z) meaning 'x blocks the betweenness of y and z', 'x tangentially blocks the betweenness of y and z' and 'x overlappingly blocks the betweenness of y and z', respectively. Note that many relation symbols are "overloaded" by using the same symbol twice with different number of arguments, as already noted above.

(A3) $\forall xy \exists z \forall uv[\beta(z, u, v) \leftrightarrow \exists w[\mathrm{P}(w, x, y) \wedge \beta(w, u, v)]]$

(A3) together with the identity axiom (A2) gives a unique region BW(x,y), the betweenness of x and y for every x and y.

(D18) $\mathrm{CX}(x) \equiv_d \forall yzw[\mathrm{P}(y, x) \wedge \mathrm{P}(z, x) \wedge \mathrm{P}(w, y, z) \rightarrow \mathrm{P}(w, x)]$

CX(x) holds true whenever x is convex.

(A4) $\forall x \exists z \forall uv[\beta(z, u, v) \leftrightarrow \exists w[\mathrm{CX}(w) \wedge \mathrm{P}(x, w) \wedge \forall y[\mathrm{P}(x, y) \wedge \mathrm{CX}(y) \rightarrow \mathrm{P}(w, y)] \wedge \beta(w, u, v)]]$

(A4) together with the identity axiom (A2) gives a unique region CO(x), the convex-hull of x for every x.

(A5) $\forall xy[\mathrm{TP}(\mathrm{CLO}(x \oplus y), \mathrm{CLO}(\mathrm{BTW}(x, y)))]$
(A6) $\exists x \forall yz[\beta(x, y, z)]$

(A5) says that every region x is a tangential part of betweenness of x and any other region. (A6) together with the identity axiom (A2) gives a unique region $\mathcal{U}$, the universe.

(A7) $\forall xyzw[\mathrm{P}(x, w, z) \wedge \mathrm{P}(y, w, z) \rightarrow [\mathrm{P}(\mathrm{BW}(x, y), \mathrm{BW}(w, z))]]$

(A7) axiomatizes the transitivity of betweenness.

(A8) $\forall xy \exists z \forall uv[\beta(z, u, v) \leftrightarrow \beta(x, u, v) \vee \beta(y, u, v)]$

(A8) together with the identity axiom (A2) gives a unique region $x \oplus y$, the sum of x and y for every x and y.

(A9) $\forall xy[\mathrm{O}(x, y) \rightarrow \exists z \forall uv[\beta(z, u, v) \leftrightarrow \exists q[\mathrm{P}(q, x) \wedge \mathrm{P}(q, y) \wedge \beta(q, u, v)]]]$

(A9) together with the identity axiom (A2) gives a unique region $x \otimes y$, the product of x and y for every x and y.

(A10) $\forall x \exists pq[\neg \beta(x, p, q) \rightarrow \exists z \forall uv[\beta(z, u, v) \leftrightarrow \exists w[\mathrm{DC}(w, x) \wedge \beta(w, u, v)]]]$

(A10) together with the identity axiom (A2) gives a unique region $-x$, the complement of x for every x such that $x \neq \mathcal{U}$.

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(A11) $\forall x \exists w \forall yz[\beta(w,y,z) \leftrightarrow \exists q[\mathrm{NTP}(q,x) \wedge \beta(q,y,z)]]$
(A11) together with the identity axiom (A2) gives a unique region $\mathrm{INT}(x)$, the interior of x for every x.

(D19) $\mathrm{CLO}(x) =_d -\mathrm{INT}(-x)$ (D20) $\mathrm{CLO}(\mathcal{U}) =_d \mathcal{U}$

(D21) $\mathrm{OP}(x) \equiv_d x = \mathrm{INT}(x)$ (D22) $\mathrm{CL}(x) \equiv_d x = \mathrm{CLO}(x)$
(D19),(D21) and (D22) define the topological operator of closure of x, $\mathrm{CLO}(x)$, and the topological properties of open and closed $\mathrm{OP}(x)$ and $\mathrm{CL}(x)$, respectively. (D20) turns the closure into a function.

(A12) $\forall xy[\mathrm{OP}(x) \wedge \mathrm{OP}(y) \wedge \mathrm{O}(x,y) \rightarrow \mathrm{OP}(x \otimes y)]$

(A13) $\forall xy[\mathrm{CX}(x) \wedge \mathrm{CX}(y) \wedge \mathrm{O}(x,y) \rightarrow \mathrm{CX}(x \otimes y)]$
(A12) and (A13) state the same thing that the product of any open and convex sets is open and convex, respectively.

(A14) $\forall x \exists y[\mathrm{CX}(y) \wedge \mathrm{P}(y,x) \wedge \neg \mathrm{P}(x,y)]$

(A15) $\exists xy[\mathrm{EC}(x,y)]$

(A16) $\exists xyz[\mathrm{DC}(x,y) \wedge \mathrm{DC}(x,z) \wedge \mathrm{DC}(z,y) \wedge \beta(x,y,z)]$

(A17) $\forall x[\mathrm{CX}(x) \rightarrow \mathrm{CX}(\mathrm{INT}(x)) \wedge \mathrm{CX}(\mathrm{CLO}(x))]$
(A14),(A15),(A16) and (A17) correspond to the model conditions of (M8),(M9),(M10) and (M11), respectively.

So far, I have given a first-order axiomatic formalism from which a proof system can be obtained by adding the axioms and rules of inference of first-order logic: Modus Ponens and Generalization. I will call this proof system as $P_{\mathcal{B}}$ and adapt the notation $\vdash_{\mathcal{B}} \phi$ iff there is a proof of the formulae ϕ in $P_{\mathcal{B}}$. Next, I present an interpretation for the language $\mathcal{L}$.

Definition 2.1. A family $\mathcal{C}$ of subsets of a set X is called a convexity on X if the following are satisfied [4]:

(CS1) $\emptyset, X \in \mathcal{C}$

(CS2) $\bigcap A \in \mathcal{C}$ for nonempty $A \subset \mathcal{C}$

(CS3) $\bigcup A \in \mathcal{C}$ whenever $A \subset \mathcal{C}$ is a chain with respect to the inclusion.

Definition 2.2. Let X be a set and $\mathcal{C}$ a convexity on X. Let X be also equipped with a topology $\mathcal{T}$. The triple $\langle X, \mathcal{T}, \mathcal{C}\rangle$ is called a topological convexity space over the domain X. We define the following set properties and operators: open(x), close(x), x°, $\overline{x}$, $\partial(x)$, $\circledast(x)$ and $\circledcirc(x)$: x is open, x is closed, interior of x, closure of x, boundary of x, convex-hull of x and x is convex, respectively. We also define $x \cap' y = x \cap y \cap \overline{(x \cap y)^\circ}$ and $x \cup' y = x \cup y \cup (\overline{x \cup y})^\circ$.

Definition 2.3. Let X be a set and $TC = \langle X, \mathcal{T}, \mathcal{C}\rangle$ be the topological convexity space over X. Let $Y \subseteq \mathcal{P}(X)$. We define the structure $\mathcal{M}_{TC} = \langle Y, \between, {}^{\delta(a)}\rangle$ such that,

(M1) $X \in Y$ (M2) $\forall x \in Y[x^\circ \in Y \wedge x^\circ \neq \emptyset \wedge x^\circ = \overline{x}^\circ]$

(M3) $\forall x \in Y[\overline{x} \in Y \wedge \overline{x} = \overline{x^\circ}\,]$ (M4) $\forall x \in Y[\circledast(x) \in Y]$

(M5) $\forall x \in Y[(\sim x)^\circ \neq \emptyset \rightarrow \sim x \in Y]$

(M6) $\forall xy \in Y[(x \cap y)^\circ \neq \emptyset \rightarrow x \cap' y \in Y]$

(M7) $\forall x \in Y \exists y \in Y[y \subset x \wedge \circledcirc(y)]$ (M8) $\forall xy \in Y[x \cup' y \in Y]$

(M9) $\exists xy \in Y[x \cap y \neq \emptyset \wedge (x \cap y)^\circ = \emptyset]$

(M10) $\exists xyz \in Y[x \cap y = \emptyset \wedge x \cap z = \emptyset \wedge z \cap y = \emptyset \wedge x \cap (\circledast(y \cup z)) \neq \emptyset]$ (M11) $\forall x \in Y[\circledcirc(x) \rightarrow \circledcirc(x^\circ) \wedge \circledcirc(\overline{x})]$

are satisfied whereas $\between$ is a function $\between$: $Y \times Y \rightarrow Y$ such that $\between(x,y) = \circledast(x \cup y)$ and $\between(x,y)$ is called the betweenness of x and y. Finally, ${}^{\delta(a)}$ is a denotation function assigning the terms in $\mathcal{L}$ to the elements of Y for a given assignment a of free-occurring variables in terms, to the elements of Y. The denotation of constants in Y is simply obtained by the function ${}^{\delta}$. The truth-relation is defined as:

$$\mathcal{M}_{TC} \models_a \beta(x,y,z) \text{ if and only if } x^{\delta(a)} \cap \between(y^{\delta(a)}, z^{\delta(a)}) \neq \emptyset$$

Naturally, we write $\models_{TC}$ for validity in every TC-model.

3 SOUNDNESS & COMPLETENESS

The proof of the following theorem can be given by an induction on the length of the proof of ϕ in $P_{\mathcal{B}}$. Base-case amounts to show that all of the axioms (A1)-(A17) hold in $\mathcal{M}_{TC}$ for every TC.

Theorem 3.1 (Soundness). *For any ϕ, if $\vdash_{\mathcal{B}} \phi$ then $\models_{TC} \phi$.*

The proof of the completeness follows from an application of the Henkin-method which consists of the following tree lemmas.

Lemma 3.2 (Lindenbaum Lemma). *Every $\mathcal{B}$-consistent set of sentences can be extended to a maximal $\mathcal{B}$-consistent set of sentences.*

Lemma 3.3 (Witness Lemma). *Every $\mathcal{B}$-consistent set of sentences Γ in $\mathcal{L}$ can be extended to a $\mathcal{B}$-consistent set of sentences Γ' in $\mathcal{L}'$ such that $\mathcal{L}' = \mathcal{L} \cup C$, $\mathcal{L} \cap C = \emptyset$ and C is an infinite set of constants which are witnesses for Γ'.*

Lemma 3.4 (Henkin Lemma). *Every maximal $\mathcal{B}$-consistent set of sentences Γ which has a set of witnesses in $\mathcal{L}$ yields a $\mathcal{M}_\Gamma$ such that $\mathcal{M}_\Gamma \models \phi$ if and only if $\phi \in \Gamma$.*

The last step before we show the completeness is to show that the structure $\mathcal{M}_\Gamma$ given by the lemma 3.4 is a $\mathcal{M}_{TC}$-model for some TC. This can be shown by the following two lemmas.

Lemma 3.5. *Let $TC_\Gamma = \langle X_\Gamma, \mathcal{T}_\Gamma, \mathcal{C}_\Gamma\rangle$ be a structure such that $X_\Gamma = \cup\{\Pi_c \mid c \in C\}$, $\mathcal{T}_\Gamma = \{\emptyset\} \cup \{\Pi_c \mid c \in C \wedge \Gamma \vdash \mathrm{OP}(c)\} \cup \{\cup s \mid s \subseteq \{\Pi_c \mid c \in C \wedge \Gamma \vdash \mathrm{OP}(c)\}\}$ and $\mathcal{C}_\Gamma = \{\emptyset\} \cup \{\Pi_c \mid c \in C \wedge \Gamma \vdash \mathrm{CX}(c)\} \cup \{\cup s \mid s \subseteq \{\Pi_c \mid c \in C \wedge \Gamma \vdash \mathrm{CX}(c)\} \wedge$ s is a chain w.r.t. inclusion$\}$ then the structure TC_Γ is a topological convexity space.*

Lemma 3.6. *The structure $\mathcal{M}_\Gamma = \langle \mathcal{D}_\Gamma, \between_\Gamma, \delta_\Gamma\rangle$ given by the lemma 3.4 is a $\mathcal{M}_{TC_\Gamma}$-model.*

Finally, the goal theorem is achieved:

Theorem 3.7 (Completeness). *For any ϕ, if $\models_{TC} \phi$ then $\vdash_{\mathcal{B}} \phi$.*

4 CONCLUSION & FURTHER WORK

I have given a first-order formalism for region-based betweenness. Moreover, I have shown that $\mathcal{B}$ is sound and complete. Although I have carried out a fundamental task regarding the properties of $\mathcal{B}$, there remains quite a bit to investigate. First of all, expressive power of this formalism could be explored, specifically in the case where orientational or positional aspects are considered. Secondly, one could seek answers to the questions regarding the computational properties of the theory.

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